Generalized Eulerian Coordinates for Relativistic Fluids: Hamiltonian Rest-Frame Instant Form, Relative Variables, Rotational Kinematics.

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Abstract

We study the rest-frame instant form of a new formulation of relativistic perfect fluids in terms of new generalized Eulerian configuration coordinates. After the separation of the relativistic center of mass from the relative variables on the Wigner hyper-planes, we define orientational and shape variables for the fluid, viewed as a relativistic extended deformable body, by introducing dynamical body frames. Finally we define Dixon's multipoles for the fluid.

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I. INTRODUCTION

Both theoretical and numerical investigations concerning relativistic hydrodynamics are becoming very important for astrophysics, cosmology and also for heavy-ions collisions. Therefore it is important to develop analytical methods able to describe the various aspects of the theory of relativistic perfect fluids, in particular their properties as isolated extended relativistic systems.

Usually relativistic fluids are described by assigning i) a unit four-velocity field $\tilde{U}^\mu(z)$ $[\tilde{U}^\mu\, \tilde{U}_\mu = 1]$; ii) the *local* thermodynamical functions *internal energy* $\tilde{\rho}(z)$, *particle density* $\tilde{n}(z)$, *local pressure* $\tilde{p}(z)$, *local temperature* $\tilde{T}(z)$, *entropy per particle* $\tilde{s}(z)$); iii) an equation of state. Here z^μ are the coordinates of a fixed point in Minkowski space-time and sometimes this description is named Eulerian. However some clarification is needed about this terminology at the relativistic level, where a notion of simultaneity has to be introduced to give sense to the relativistic equations of motion, which very often are deduced from an action principle.

In the standard approach to non-relativistic fluid dynamics, not based on variational principles, two distinct points of view are usually used [1]: the *Lagrangian (or material)* point of view and the *Eulerian (or local)* point of view. The equations of motion, i) the continuity equation or mass conservation; ii) Euler-Newton equations or balance of linear momentum; iii) the conservation of energy; look different in the two points of view.

A) In the *Lagrangian point of view* the fluid is described by the *flux lines* $\vec{x}(t, \vec{x}_o)$ with $\vec{x}(0, \vec{x}_o) = \vec{x}_o$, defined as the integral lines of the 3-velocity field $\vec{u}(t, \vec{x}_o) = \frac{\partial \vec{x}(t, \vec{x}_o)}{\partial t}$. Each integral line is labeled with its initial coordinate $\vec{x}_o$. The coordinates $\vec{x}_o$'s are the *Lagrangian (or comoving) coordinates of the Lagrangian point of view*. The flux lines have the role to describe the *mechanical* aspect of the flow of the fluid. If we think that associated with each flux line there is a *material particle* defined by an infinitesimal volume of fluid around the point $\vec{x}_o$, then the flux line $\vec{x}(t, \vec{x}_o)$ is also the trajectory followed by the material particle, so that also the name *material point of view* is used. For a fixed value of $\vec{x}_o$, $\vec{x}(t, \vec{x}_o)$ specifies the path of the mass element which was at $\vec{x}_o$ at $t = 0$; for a fixed value of t, $\vec{x}(t, \vec{x}_o)$ determines the transformation of the region initially occupied by the whole mass of the fluid. For every

[1]See for instance Section 3.2 of Ref. [1].

local thermodynamical function evaluated on the flux lines $\tilde{\mathcal{G}}(t, \vec{x}(t, \vec{x}_o))$ its expression in the Lagrangian point of view is $\hat{\mathcal{G}}(t, \vec{x}_o) = \tilde{\mathcal{G}}(t, \vec{x}(t, \vec{x}_o))$. Therefore the equations of motion in the Lagrangian point of view are written using only the ordinary time derivative, $\frac{\partial}{\partial t} \hat{\mathcal{G}}(t, \vec{x}_o)$.

B) Instead in the *Eulerian (or local) point of view* the fluid is described by taking as *Eulerian coordinates* a set of coordinates $\vec{x}$ referring to a fixed location in space and not to a moving mass element of fluid and by using the 3-velocity field $\vec{v}(t, \vec{x})$, which is the velocity of the fluid particle that happens to be at the location $\vec{x}$ at time t. If at time t we make the identification $\vec{x} = \vec{x}(t, \vec{x}_o)$, namely the fixed coordinate is seen as the coordinate of the flux line through that point, we recover the 3-velocity field of the other point of view: $\vec{v}(t, \vec{x})|_{\vec{x}=\vec{x}(t,\vec{x}_o)} = \vec{u}(t, \vec{x}_o)$. Now a local thermodynamical function is described by the local function $\tilde{\mathcal{G}}(t, \vec{x})$ with $\tilde{\mathcal{G}}(t, \vec{x})|_{\vec{x}=\vec{x}(t,\vec{x}_o)} = \hat{\mathcal{G}}(t, \vec{x}_o)$, so that this description of the fluid is also named the *local point of view*. The equations of motion in the Eulerian point of view involve the so-called *total (or material) derivative* $\frac{D}{Dt} \tilde{\mathcal{G}}(t, \vec{x}) = (\frac{\partial}{\partial t} + \vec{u}(t, \vec{x}) \cdot \frac{\partial}{\partial \vec{x}}) \tilde{\mathcal{G}}(t, \vec{x})$, since we have $\frac{\partial}{\partial t} \hat{\mathcal{G}}(t, \vec{x}_o) = \frac{\partial}{\partial t} \tilde{\mathcal{G}}(t, \vec{x}(t, \vec{x}_o)) = \Big(\frac{\partial}{\partial t} + \vec{u}(t, \vec{x}_o) \cdot \frac{\partial}{\partial \vec{x}}\Big) \tilde{\mathcal{G}}(t, \vec{x}(t, \vec{x}_o)) = \Big(\frac{D}{Dt} \tilde{\mathcal{G}}(t, \vec{x})\Big)|_{\vec{x}=\vec{x}(t,\vec{x}_o)}$.

To extend these descriptions to the relativistic level in the Minkowski space-time M^4 with coordinates z^μ without introducing an explicit breaking of covariance like the one implied by the decomposition $z^\mu = (z^o = ct; \vec{z})$, it is convenient to work in the context of *Dirac's parametrized Minkowski theories* [2,3] on arbitrary (simultaneity and Cauchy) space-like hyper-surfaces, leaves of the foliation associated to a 3+1 splitting of Minkowski space-time. If τ is the scalar parameter (mathematical time) which labels the leaves Σ_τ of the foliation, $\vec{\sigma}$ are curvilinear coordinates on them (with respect to an arbitrary centroid $x^\mu(\tau) = z^\mu(\tau, \vec{0})$ chosen as origin) and $z^\mu(\tau, \vec{\sigma})$ are the embeddings of the hyper-surfaces Σ_τ in Minkowski space-time, then every local thermodynamical function has an *equal time* re-formulation: $\tilde{\mathcal{G}}(z(\tau, \vec{\sigma})) = \mathcal{G}(\tau, \vec{\sigma})$. By using these adapted coordinates we have the following natural relativistic generalization of the two points of view.

A) In the *Lagrangian (or comoving) point of view* the fluid is described by the flux lines $\tilde{\zeta}^\mu(z_o, \tilde{\tau})$, defined as the integral lines of the four-velocity field with *initial condition* z^μ_o and parametrized by their proper time $\tilde{\tau}$

$$\frac{d}{d\tilde{\tau}} \tilde{\zeta}^\mu(z_o, \tilde{\tau}) = \tilde{U}^\mu(\tilde{\zeta}^\mu(z_o, \tilde{\tau})), \qquad \tilde{\zeta}^\mu(z_o, 0) = z^\mu_o. \tag{1.1}$$

The points z^μ_o used for the initial conditions must belong to a *space-like Cauchy hyper-surface* in Minkowski space-time. If we use the embeddings $z^\mu(\tau, \vec{\sigma})$ of the hyper-surfaces Σ_τ

of a foliation, the flux world-lines are described by functions $\zeta^\mu(\tau,\vec{\sigma}_o) = \tilde{\zeta}^\mu(z_o, \tilde{\tau}(\tau,\vec{\sigma}_o)) = z^\mu(\tau, \vec{\Sigma}(\tau,\vec{\sigma}_o))$ with $\tilde{\tau}(0,\vec{\sigma}_o) = 0$, $\vec{\Sigma}(0,\vec{\sigma}_o) = \vec{\sigma}_o$, $\zeta^\mu(0,\vec{\sigma}_o) = z^\mu(\tau_o,\vec{\sigma}_o) = z^\mu_o$. Since τ is not the proper time of any flux line, on Σ_τ we have $\frac{d\zeta^\mu(\tau,\vec{\sigma}_o)}{d\tau} / \sqrt{(\frac{d\zeta(\tau,\vec{\sigma}_o)}{d\tau})^2} = U^\mu(\tau,\vec{\sigma}_o)$ for the flux line through $\vec{\sigma}_o$ at $\tau = 0$.

In these adapted coordinates the coordinates $\vec{\sigma}_o$ on the Cauchy surface Σ_{τ_o} are the *Lagrangian (or comoving) coordinates of the Lagrangian point of view*, replacing the non-relativistic $\vec{x}_o$, while the functions $\vec{\Sigma}(\tau,\vec{\sigma}_o)$, describing the flux lines, replace the non-relativistic $\vec{x}(t,\vec{x}_o)$ and $\vec{u}(\tau,\vec{\sigma}_o) = \frac{\partial\vec{\Sigma}(\tau,\vec{\sigma}_o)}{\partial\tau}$ is the 3-velocity field replacing $\vec{u}(t,\vec{x}_o)$. The local thermodynamical functions evaluated along the flux lines $\tilde{\mathcal{G}}(\zeta^\mu(\tau,\vec{\sigma}_o)) = \tilde{\mathcal{G}}(z^\mu(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))) = \mathcal{G}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))$ have the expression $\hat{\mathcal{G}}(\tau,\vec{\sigma}_o) = \mathcal{G}(\tau,\vec{\sigma})|_{\vec{\sigma}=\vec{\Sigma}(\tau,\vec{\sigma}_o)} = \mathcal{G}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))$. The equations of motion involve only the ordinary time derivative $\frac{\partial}{\partial\tau}\hat{\mathcal{G}}(\tau,\vec{\sigma}_o)$. To each flux line there is associated a material particle at $\vec{\sigma}_o$ on the Cauchy surface $\Sigma_{\tau=0}$.

B) In the *Eulerian point of view* the fluid is described by taking the fixed 3-coordinates $\vec{\sigma}$ (replacing the non-relativistic $\vec{x}$) as *Eulerian coordinates* and τ as the time in the coordinatization dictated by the embedding. The 3-velocity field is $\vec{v}(\tau,\vec{\sigma})$ and it can be shown that its expression in terms of the 4-velocity field $U^A(\tau,\vec{\sigma})$, written in adapted coordinates, is $\vec{v}(\tau,\vec{\sigma}) = \vec{U}(\tau,\vec{\sigma})/U^\tau(\tau,\vec{\sigma})$ [see Eq.(3.16)]. When the Eulerian coordinate is identified with the flux line $\zeta^\mu(\tau,\vec{\sigma}_o) = z^\mu(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))$ we get $\vec{v}(\tau,\vec{\sigma})|_{\vec{\sigma}=\vec{\Sigma}(\tau,\vec{\sigma}_o)} = \frac{\partial\vec{\Sigma}(\tau,\vec{\sigma}_o)}{\partial\tau}$. The local thermodynamical functions are described by local functions $\mathcal{G}(\tau,\vec{\sigma})$ with $\mathcal{G}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o)) = \hat{\mathcal{G}}(\tau,\vec{\sigma}_o)$. Their equations of motion in the Eulerian point of view involve a total τ-derivative, $\frac{D}{D\tau}\mathcal{G}(\tau,\vec{\sigma}) = (\frac{\partial}{\partial\tau} + \vec{v}(\tau,\vec{\sigma})\cdot\frac{\partial}{\partial\vec{\sigma}})\,\mathcal{G}(\tau,\vec{\sigma})$, since $\frac{\partial}{\partial\tau}\hat{\mathcal{G}}(\tau,\vec{\sigma}_o) = (\frac{\partial}{\partial\tau} + \frac{\partial\vec{\Sigma}(\tau,\vec{\sigma}_o)}{\partial\tau}\cdot\frac{\partial}{\partial\vec{\sigma}})\,\mathcal{G}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o)) = \Big(\frac{D}{D\tau}\mathcal{G}(\tau,\vec{\sigma})\Big)|_{\vec{\sigma}=\vec{\Sigma}(\tau,\vec{\sigma}_o)}$.

Therefore, while in the Lagrangian (or material) point of view we follow the evolution of a thermodynamic function evaluated in a material particle (namely along the flux line physically determined by the average particle motion), in the Eulerian (or local) point of view we follow the evolution of the same function evaluated in a given space-time point coinciding with the associated material particle only on the Cauchy surface.

Till now only in the non-relativistic framework of the Euler-Newton equations there has been a study of the transformation from Eulerian to Lagrangian coordinates [4].

The next problem is how to derive the equations of motion of the Lagrangian and Eulerian point of views as the Euler-Lagrange (EL) equations of an action principle.

An important development in relativistic hydrodynamics has been given by Brown [5], who built a general framework encompassing all the known variational principles for relativistic perfect fluids and clarifying the inter-connections among very different Lagrangian approaches.

In Brown's paper the fluid is described by means of a set of scalar 3-dimensional configuration variables $\tilde{\alpha}^i(z)$, $i = 1, 2, 3$, of the space-time coordinates z^μ, interpreted as *Lagrangian (or comoving) configuration coordinates* for the fluid labeling the fluid flow lines.

In Ref. [6] one of the action principles of Ref. [5] has been re-formulated in the contest of *Dirac's parametrized Minkowski theories* [2,3] on arbitrary (simultaneity and Cauchy) space-like hyper-surfaces, leaves of the foliation associated to a 3+1 splitting of Minkowski space-time for arbitrary equations of state of the type $\tilde{\rho}(z) = \tilde{\rho}(\tilde{n}(z), \tilde{s}(z))$. Now the Lagrangian (or comoving) coordinates of the fluid are $\alpha^i(\tau, \vec{\sigma}) = \tilde{\alpha}^i(z(\tau, \vec{\sigma}))$ [2]. For each value of τ, we can invert $\alpha^i = \alpha^i(\tau, \vec{\sigma})$ to $\vec{\sigma} = \vec{\sigma}(\tau, \alpha^i)$ and use the α^i's as a special coordinate system on Σ_τ inside the fluid support $V_\alpha(\tau) \subset \Sigma_\tau$: $z^\mu(\tau, \vec{\sigma}(\tau, \alpha^i)) = \check{z}^\mu(\tau, \alpha^i)$. This approach is reviewed in Section II.

Since it can be shown [see Eqs.(2.16) and(3.2)] that we have $\alpha^i(\tau, \vec{\sigma})|_{\vec{\sigma}=\vec{\Sigma}(\tau,\vec{\sigma}_o)} = \alpha^i(0, \vec{\sigma}_o) = \alpha^i_o(\vec{\sigma}_o)$, namely that the coordinates $\alpha^i(\tau, \vec{\sigma})$ are constant along the flux lines, this explains why these coordinates are a possible set of Lagrangian (comoving) coordinates for the fluid in alternative to the $\vec{\sigma}_o$'s . Therefore every action principle studied in Ref. [5] generates Euler-Lagrange equations [like Eq.(2.19)] which describe what happens at the fixed location $\vec{\sigma}$ when τ changes, i.e. these EL equations generate the equations of motion of the thermodynamical functions $\mathcal{G}(\tau, \vec{\sigma})$ in the Eulerian (or local) point of view. This is a consequence of the necessity that the configuration variables of an action principle be τ-dependent, even when, like in this case, they are used to simulate the fixed comoving coordinates of the fluid.

In this paper we will show that it is possible to define a variational approach whose configuration variables are the adapted 3-coordinates $\vec{\Sigma}(\tau, \vec{\sigma}_o)$ of the flux lines, which evolve from

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[2]The fluid is supposed to have compact support $V_\alpha(\tau) \subset \Sigma_\tau$, whose boundary $\partial V_\alpha(\tau)$ is dynamically determined as the 2-dimensional surface in each Σ_τ where the pressure vanishes, $\tilde{p}(z(\tau, \vec{\sigma})) = 0$ for $\vec{\sigma} \in \partial V_\alpha(\tau)$. In the case of N disjoint fluid sectors, we can use the same description with $V_\alpha(\tau) = \cup_i V_{\alpha\, i}(\tau)$ till when the compact supports $V_{\alpha\, i}(\tau)$ do not overlap.

the Lagrangian coordinates $\vec{\sigma}_o = \vec{\Sigma}(0, \vec{\sigma}_o)$ on $\Sigma_{\tau=0}$, instead of the Lagrangian (or comoving) variables $\vec{\alpha}(\tau, \vec{\sigma})$ used in Ref. [5]. The configuration 3-coordinates $\vec{\Sigma}(\tau, \vec{\sigma}_o)$ are strictly speaking neither Lagrangian nor Eulerian coordinates. However they can be considered as *generalized Eulerian configuration coordinates*, because through the position $\vec{\sigma} = \vec{\Sigma}(\tau, \vec{\sigma}_o)$ (connecting the two points of view) they allow to obtain the Eulerian description of what happens in the fixed point $\vec{\sigma}_o$ when τ changes. Indeed in this case the resulting EL equations (3.17) [replacing the Eulerian ones (2.19)] will correspond to the equations of motion of the thermodynamical functions $\hat{\mathcal{G}}(\tau, \vec{\sigma}_o)$ in the Lagrangian (or material) point of view. It will be shown (see footnote 7) how we can transform these EL equations into the equations of motion in the Eulerian point of view for $\mathcal{G}(\tau, \vec{\sigma})$. This will allow to obtain a Hamiltonian formulation with Eulerian coordinates using Poisson brackets instead of the Lie-Poisson brackets of Ref. [7].

We shall study the Hamiltonian first class constraints of the fluid on arbitrary space-like hyper-surfaces and, then, the restriction to the rest frame foliation (Wigner hyper-planes), in order to obtain its *rest-frame instant form*, already used for particles and fields in Refs. [8–12]. Then, following the methods of Refs. [13–15], we shall study the problem of the center-of-mass and relative variables, the separation of relative variables in orientational and vibrational ones by means of the introduction of dynamical body frames and Dixon's multipoles [16,17] of the fluid.

In this way we get a complete control on the relativistic kinematics of perfect fluids considered as extended deformable objects. The next step will be to couple the perfect fluid to metric and tetrad canonical gravity, whose rest-frame instant form has been developed in Refs. [18–20], both to develop a scheme of Hamiltonian numerical gravity in accord with constraint theory and to study the linearized theory in a completely fixed 3-orthogonal Hamiltonian gauge following the scheme of Ref. [21]. Another future development [22] will be to study the non-relativistic limit of this approach and, then, after the addition of Newton gravitational potential, to recover the ellipsoidal equilibrium configurations [23] in this kinematics in the case of incompressible fluids.

While in Section II we review the description of perfect fluids with Lagrangian coordinates, in Section III we introduce the new formulation with Eulerian coordinates. In Section IV we derive the Hamiltonian formulation associated with the action written in the previous Section and we get the usual constraints of *Dirac's parametrized Minkowski theories* and their Dirac Poisson algebra [2]. In Section V the *rest frame instant form* of the dynamic is constructed. This form of the dynamic is such that we can discuss the problem of the separation of the relative variables from the center of mass-like variables and the analogous problem for the rotational and shape variables in the same way as it has already been done for relativistic particles. This is done in Sections VI and VII. In Section VIII we discuss the various either exact or approximate forms in which the invariant mass of the fluid, i.e. the Hamiltonian in the rest-frame instant form, may be presented as a function of the orientational and shape variables. Dixon's multipoles for the fluid are defined in Section IX. In the final Section there are some concluding remarks.

Appendix A reviews notations on space-like hyper-surfaces. In Appendix B there is a list of the equations of state for which we can obtain a closed form of the fluid invariant mass. Appendix C contains remarks on Poisson brackets. Appendix D describes the Gartenhaus-Schwartz transformation. Finally in Appendix E there are some solutions for the kernels associated with the relative variables.

II. LAGRANGIAN FORMULATION OF THE DYNAMICS OF RELATIVISTIC PERFECT FLUIDS IN PARAMETRIZED MINKOWSKI SPACE-TIME WITH LAGRANGIAN COMOVING CONFIGURATION COORDINATES.

In this Section we review some of the results of Refs. [5,6]. One of the many action principles for a Lagrangian description of relativistic perfect fluid dynamics described in Ref. [5] has been re-formulated in Ref. [6] in the context of the *parametrized Minkowski theories* [8,3]. As said in the Introduction the starting point of these theories [2] is the foliation of the Minkowski space-time by a family of space-like hyper-surfaces Σ_τ defined by the *embedding* $z^\mu(\tau,\vec{\sigma})$ $(R\times\Sigma\rightarrow M^4)$ [3]. The fields $z^\mu(\tau,\vec{\sigma})$ define a coordinates transformation $z^\mu\mapsto\sigma^{\check{A}}$ on the pseudo-Riemannian manifold M^4, a field of *cotetrads* (μ are the flat indices)

$$z^\mu_{\check{A}}(\tau,\vec{\sigma})=\frac{\partial z^\mu(\tau,\vec{\sigma})}{\partial\sigma^{\check{A}}}, \tag{2.1}$$

and the induced metric

$$g_{\check{A}\check{B}}(\tau,\vec{\sigma})=z^\mu_{\check{A}}(\tau,\vec{\sigma})\eta_{\mu\nu}z^\nu_{\check{B}}(\tau,\vec{\sigma}). \tag{2.2}$$

See Appendix A for other properties of space-like hyper-surfaces.

In parametrized Minkowski theories the Lagrangian of every isolated system is written as a functional of the Lagrangian coordinates $\alpha^i(\tau,\vec{\sigma})$ of the system, adapted to the foliation, and of the embedding $z^\mu(\tau,\vec{\sigma})$ interpreted as the Lagrangian coordinates describing the hyper-surface in this enlarged configuration space. This functional is determined by considering the Lagrangian of the system coupled to an external gravitational field and replacing the 4-metric $g_{\mu\nu}(z)$ with the induced metric (2.2) in the adapted coordinates.

As said in the Introduction in the Eulerian point of view, the relativistic perfect fluid is characterized by the 4-velocity field $\tilde{U}^\mu(z)$ $(\tilde{U}_\mu\tilde{U}^\mu=1)$ on Minkowski space-time M^4 and by a set of local thermodynamical functions. After the foliation of M^4 with the hyper-surfaces Σ_τ, the 4-velocity field has the adapted covariant components

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[3] We use the notation $\sigma^{\check{A}}=(\tau,\sigma^{\check{r}})$ where $\check{A}=(\tau,\check{r})$, $\check{r}=1,2,3$, for these coordinates adapted to the foliation. The notation r will be reserved in Section V for the 3-vectors on the *Wigner hyper-planes*

$$U_{\check{A}}(\tau,\vec{\sigma}) = z^{\mu}_{\check{A}}(\tau,\vec{\sigma})\tilde{U}_{\mu}(z(\tau,\vec{\sigma})). \tag{2.3}$$

With the adopted parametrization of Minkowski space-time the local thermodynamical functions can be seen as functions of $(\tau,\vec{\sigma})$ by means of the replacement $z^{\mu} = z^{\mu}(\tau,\vec{\sigma})$ [4] . In particular let us consider the *numerical density of particles* $n(\tau,\vec{\sigma})$. Together with the 4-velocity field $U^{\check{A}}(\tau,\vec{\sigma})$ it defines the *numerical density current* $n(\tau,\vec{\sigma})\,U^{\check{A}}(\tau,\vec{\sigma})$. *The conservation of the total particle number* is the following constraint on this current (";" and "," denote the covariant and ordinary derivative, respectively)

$$\Big[n(\tau,\vec{\sigma})\,U^{\check{A}}(\tau,\vec{\sigma})\Big]_{;\check{A}} = \frac{1}{\sqrt{g(\tau,\vec{\sigma})}}\,\frac{\partial}{\partial\sigma^{\check{A}}}\left[\sqrt{g(\tau,\vec{\sigma})}\,n(\tau,\vec{\sigma})\,U^{\check{A}}(\tau,\vec{\sigma})\right] = 0. \tag{2.4}$$

We can also to consider the *energy density function* $\rho(\tau,\vec{\sigma})$, the local *pressure* $p(\tau,\vec{\sigma})$, the local *temperature* $T(\tau,\vec{\sigma})$, the *entropy per particles* $s(\tau,\vec{\sigma})$ and the *chemical potential*

$$\mu(\tau,\vec{\sigma}) = \frac{\rho(\tau,\vec{\sigma}) + p(\tau,\vec{\sigma})}{n(\tau,\vec{\sigma})}. \tag{2.5}$$

The *first principle of thermodynamics* is given by the following differential relation

$$d\rho(\tau,\vec{\sigma}) = \mu(\tau,\vec{\sigma})\,dn(\tau,\vec{\sigma}) + n(\tau,\vec{\sigma})\,T(\tau,\vec{\sigma})\,ds(\tau,\vec{\sigma}). \tag{2.6}$$

The *equation of state* is given interpreting ρ as a function of n and s

$$\rho(\tau,\vec{\sigma}) = \rho(n(\tau,\vec{\sigma}),s(\tau,\vec{\sigma})), \tag{2.7}$$

and we can obtain the other thermodynamical quantities as functions of n and s, in particular

$$T(\tau,\vec{\sigma}) \equiv \frac{1}{n(\tau,\vec{\sigma})}\,\frac{\partial\rho}{\partial s}(n(\tau,\vec{\sigma}),s(\tau,\vec{\sigma})),$$

$$p(\tau,\vec{\sigma}) \equiv n(\tau,\vec{\sigma})\,\frac{\partial\rho}{\partial n}(n(\tau,\vec{\sigma}),s(\tau,\vec{\sigma})) - \rho(n(\tau,\vec{\sigma}),s(\tau,\vec{\sigma})). \tag{2.8}$$

Finally, we have to add *the entropy conservation*

$$U^{\check{A}}(\tau,\vec{\sigma})\,\frac{\partial}{\partial\sigma^{\check{A}}}\,s(\tau,\vec{\sigma}) = 0. \tag{2.9}$$

This constraint tells us that we don't have loss of entropy out the flux tube defined by the fluid's flux lines.

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[4]From now on we shall denote with $f(\tau,\vec{\sigma})$ the functions $f(\tau,\vec{\sigma}) = \tilde{f}(z(\tau,\vec{\sigma}))$.

Due to the constraints (2.4),(2.9), the four-velocity field $U_{\check{A}}(\tau,\vec{\sigma})$ and the independent thermodynamic functions, $n(\tau,\vec{\sigma})$, $s(\tau,\vec{\sigma})$ are a set of redundant variables for the fluid description . In Ref. [5] it is shown that the constraints (2.4),(2.9) may be enforced introducing some *Lagrangian (or comoving) variables* $\alpha^i(\tau,\vec{\sigma})$ for describing the fluid. It is useful to introduce the fields $J^{\check{A}}(\tau,\vec{\sigma})$

$$\sqrt{g(\tau,\vec{\sigma})}\, n(\tau,\vec{\sigma})\, U^{\check{A}}(\tau,\vec{\sigma}) = J^{\check{A}}(\tau,\vec{\sigma}). \tag{2.10}$$

These fields are dependent on the Lagrangian coordinates $\alpha^i(\tau,\vec{\sigma}),\ i=1,2,3$, according to the definition

$$\begin{aligned} J^{\tau}(\tau,\vec{\sigma}) &= -\det\, I(\tau,\vec{\sigma}),\\ J^{\check{r}}(\tau,\vec{\sigma}) &= \det\, I(\tau,\vec{\sigma})\ \Big[I^{-1}(\tau,\vec{\sigma})\Big]^{\check{r}}_{i}\ \frac{\partial\alpha^i(\tau,\vec{\sigma})}{\partial\tau}, \end{aligned} \tag{2.11}$$

where the 3×3 matrix $I(\tau,\vec{\sigma})$ is

$$[I(\tau,\vec{\sigma})]^i_{\check{r}} = \left(\frac{\partial\alpha^i(\tau,\vec{\sigma})}{\partial\sigma^{\check{r}}}\right). \tag{2.12}$$

From the definition (2.11) it follows that ($\sigma^\tau \equiv \tau$)

$$\frac{\partial}{\partial\sigma^{\check{A}}}\, J^{\check{A}}(\tau,\vec{\sigma}) = 0. \tag{2.13}$$

This equation is equivalent to the constraint (2.4). In particular, it follows from Eq. (2.13) that the *total number of particles* on the hyper-surfaces Σ_τ may be defined as ($V_\alpha(\tau)$ is the fluid's volume on Σ_τ)

$$\mathcal{N} = \int_{V_\alpha(\tau)} d^3\sigma\, J^\tau(\tau,\vec{\sigma}), \tag{2.14}$$

and it is conserved by the evolution in the time parameter τ. In this parametrization the entropy per particle is a function of the $\alpha^i(\tau,\vec{\sigma})$ alone

$$s \equiv s(\alpha^i(\tau,\vec{\sigma})). \tag{2.15}$$

By construction it satisfy the entropy constraint (2.9) [see also the following equation (2.16)].

The Lagrangian coordinates $\alpha^i(\tau,\vec{\sigma})$ can be interpreted as resulting from a coordinate transformation $\sigma^{\check{r}} \mapsto \alpha^i$ on the hyper-surface Σ_τ. In particular $\alpha^i(0,\vec{\sigma})$ define a coordinate transformation $\sigma^{\check{r}} \mapsto \alpha^i(0,\vec{\sigma})$ on the hyper-surface $\Sigma_{\tau=0}$; if $V_\alpha(0)$ is the fluid total volume in this hyper-surface, every point in the total volume on the hyper-surface Σ_τ, $V_\alpha(\tau)$, is in a one to one correspondence with a point in $V_\alpha(0)$ by means of the flux lines. Due to the definitions (2.10) and (2.11) it follows that the fields $\alpha^i(\tau,\vec{\sigma})$ are *constant along the flux lines*, since by construction we have

$$U^{\check{A}}(\tau,\vec{\sigma})\,\frac{\partial}{\partial\sigma^{\check{A}}}\,\alpha^i(\tau,\vec{\sigma})=0. \tag{2.16}$$

Then the fields $\alpha^i(\tau,\vec{\sigma})$ can be interpreted also as *labels* assigned to the flux lines; the field $\alpha^i(\tau,\vec{\sigma})$ tells us that the flux line labeled with α^i *goes through* the point $z^\mu(\tau,\vec{\sigma})\in M^4$.

With the previous definitions and observations, the action defined in Section 5 of Ref. [5] has been rewritten in Ref. [6] in the form

$$S=-\int d\tau\,d^3\sigma\,\sqrt{g(\tau,\vec{\sigma})}\,\rho(n[\alpha],s[\alpha]), \tag{2.17}$$

where s is given by Eq.(2.15) and from Eq.(2.10) it follows that

$$n(\tau,\vec{\sigma})=\frac{\sqrt{g_{\check{A}\check{B}}(\tau,\vec{\sigma})\,J^{\check{A}}(\tau,\vec{\sigma})\,J^{\check{B}}(\tau,\vec{\sigma})}}{\sqrt{g(\tau,\vec{\sigma})}}. \tag{2.18}$$

The stationarity of the action with respect to variations of the α's gives the fluid equations of motion [5]

$$0=2\,V_{[\check{E},\check{F}]}(\tau,\vec{\sigma})\,U^{\check{F}}(\tau,\vec{\sigma})+T(\tau,\vec{\sigma})\,s_{,\check{E}}(\tau,\vec{\sigma}), \tag{2.19}$$

where

$$V_{\check{A}}(\tau,\vec{\sigma})=\mu(\tau,\vec{\sigma})\,U_{\check{A}}(\tau,\vec{\sigma}), \tag{2.20}$$

is the *Taub vector*, if $\mu(\tau,\vec{\sigma})$ is the chemical potential (2.5). In the previous relation we have used the notation

$$V_{[\check{E},\check{F}]}=V_{[\check{E};\check{F}]}=\frac{V_{\check{E},\check{F}}-V_{\check{F},\check{E}}}{2}. \tag{2.21}$$

The variation of the action with respect to the metric variations $\delta g_{\check{A}\check{B}}$ defines the *stress-energy tensor*

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[5]They are equations of motion in the Eulerian (or local) point of view.

$$T^{\check{A}\check{B}}(\tau,\vec{\sigma}) = -\frac{2}{\sqrt{g(\tau,\vec{\sigma})}}\,\frac{\delta S}{\delta g_{\check{A}\check{B}}(\tau,\vec{\sigma})} =$$

$$= \Big[\rho(\tau,\vec{\sigma}) + p(\tau,\vec{\sigma})\Big]\, U^{\check{A}}(\tau,\vec{\sigma})\, U^{\check{B}}(\tau,\vec{\sigma}) - p(\tau,\vec{\sigma})\, g^{\check{A}\check{B}}(\tau,\vec{\sigma}). \qquad (2.22)$$

In Ref. [5] it is showed that the equations of motion (2.19) are equivalent to the stress-energy tensor conservation law

$$T^{\check{A}\check{B}}{}_{;\check{B}} = 0. \qquad (2.23)$$

To see this, we have to observe that Eq.(2.23) is equivalent to

$$\begin{aligned} U_{\check{A}}\, T^{\check{A}\check{B}}{}_{;\check{B}} &= 0,\\ (g_{\check{A}\check{B}} - U_{\check{A}}\, U_{\check{B}})\, T^{\check{B}\check{C}}{}_{;\check{C}} &= 0. \end{aligned} \qquad (2.24)$$

The first of these equations is equivalent to the entropy constraint (2.9), which is then satisfied,

$$U_{\check{A}}\, T^{\check{A}\check{B}}{}_{;\check{B}} = -n\, T\, s_{,\check{B}}\, U^{\check{B}} = 0. \qquad (2.25)$$

The second equation is equivalent to the *Euler equations*

$$(\rho + p)\, U_{\check{A};\check{B}}\, U^{\check{B}} + (-\delta^{\check{B}}_{\check{A}} + U^{\check{A}}\, U^{\check{B}})\, p_{,\check{B}} = 0. \qquad (2.26)$$

The stationarity of the action (2.17) with respect to the variations of the $z^\mu(\tau,\vec{\sigma})$'s gives us the equations of motion of the embeddings $z^\mu(\tau,\vec{\sigma})$. Since the action depends on $z^\mu(\tau,\vec{\sigma})$ only through the induced metric, they are

$$\frac{\delta S}{\delta z^\mu(\tau,\vec{\sigma})} = 2\,\eta_{\mu\nu}\,\frac{\partial}{\partial \sigma^{\check{A}}}\,\left[\frac{\delta S}{\delta g_{\check{A}\check{B}}(\tau,\vec{\sigma})}\, z^\nu_{\check{B}}(\tau,\vec{\sigma})\right] =$$

$$= -\eta_{\mu\nu}\,\frac{\partial}{\partial \sigma^{\check{A}}}\,\left[\sqrt{g(\tau,\vec{\sigma})}\, T^{\check{A}\check{B}}(\tau,\vec{\sigma})\, z^\nu_{\check{B}}(\tau,\vec{\sigma})\right] =$$

$$= \sqrt{g(\tau,\vec{\sigma})}\, z^{\check{C}}_\mu(\tau,\vec{\sigma})\, g_{\check{C}\check{D}}(\tau,\vec{\sigma})\, \Big[T^{\check{D}\check{A}}(\tau,\vec{\sigma})\Big]_{;\check{A}} = 0. \qquad (2.27)$$

Due to the stress-energy conservation (2.23), following from the fluid equations of motion (2.19), these equations are always satisfied without any restriction on the embeddings $z^\mu(\tau,\vec{\sigma})$, which remain arbitrary. In other words, the equations of motion (2.19) and (2.27) are not independent. This is the Lagrangian manifestation that the parametrized Minkowski theories are singular theories. In these theories the $z^\mu(\tau, \vec{\sigma})$ are *gauge variables* and in the Hamiltonian formulation their conjugate momenta are defined by first class Dirac constraints [3,8] as it will shown in Section IV .

III. LAGRANGIAN FORMULATION OF THE DYNAMICS OF RELATIVISTIC PERFECT FLUIDS IN PARAMETRIZED MINKOWSKI SPACE-TIME WITH EULERIAN CONFIGURATION COORDINATES.

In this Section we introduce a different parametrization of the action (2.17) using a new set of configuration coordinates. The geometrical interpretation of the *old* Lagrangian (comoving) coordinates $\alpha^i(\tau,\vec{\sigma})$ as labels for the flux lines suggests that it is possible to parametrize the action (2.17) with new adapted 3-coordinates $\vec{\Sigma}(\tau,\vec{\sigma}_o)$, which describe the flux lines as the integral curves of the 4-velocity field starting from the Lagrangian coordinates $\vec{\sigma}_o = \vec{\Sigma}(0,\vec{\sigma}_o)$. As explained in the Introduction they can be named generalized Eulerian configuration coordinates. The derived variables $\zeta^\mu(\tau,\vec{\sigma}_o) = z^\mu(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))$ define the four-dimensional *flux line* going through $\vec{\sigma}_o$ at $\tau = 0$

$$\frac{\frac{d}{d\tau}\,\zeta^\mu(\tau,\vec{\sigma}_o)}{\sqrt{\eta_{\alpha\beta}\,\frac{d}{d\tau}\,\zeta^\alpha(\tau,\vec{\sigma}_o)\,\frac{d}{d\tau}\,\zeta^\beta(\tau,\vec{\sigma}_o)}} = \tilde{U}^\mu(\zeta(\tau,\vec{\sigma}_o)), \tag{3.1}$$

with the initial condition $\zeta^\mu(0,\vec{\sigma}_o) = z^\mu(0,\vec{\sigma}_o)$, namely $\vec{\Sigma}(0,\vec{\sigma}_o) = \vec{\sigma}_o$.

Moreover, due to Eq.(2.16), we get consistently

$$\alpha^i(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o)) = \alpha^i(0,\vec{\sigma}_o) \equiv \alpha^i_o(\vec{\sigma}_o), \tag{3.2}$$

i.e. the functions $\alpha^i_o(\vec{\sigma}_o)$ (a possible set of Lagrangian coordinates replacing the $\vec{\sigma}_o$'s) are constant along the flux line through each point $\vec{\sigma}_o$. Then the inverse function theorem implies $\vec{\Sigma}(\tau,\vec{\sigma}_o) = \vec{F}(\tau,\vec{\alpha}(0,\vec{\sigma}_o))$ and $\vec{\sigma}_o = \vec{\Sigma}(0,\vec{\sigma}_o) = \vec{F}(0,\vec{\alpha}(0,\vec{\sigma}_o))$. Therefore at $\tau = 0$ the generalized Eulerian coordinates of the fluid are just the Lagrangian 3-coordinates $\vec{\sigma}_o$ of the points where the flux lines intersect the hyper-surface $\Sigma_{\tau=0}$ and they are connected to the Lagrangian comoving coordinates $\alpha^i(0,\vec{\sigma})$ at $\tau = 0$ by the change of coordinates $\vec{\sigma} \mapsto \vec{\alpha}(0,\vec{\sigma})$. By inverting $\vec{\sigma} = \vec{\Sigma}(\tau,\vec{\sigma}_o)$ to $\vec{\sigma}_o = \vec{g}_{\vec{\Sigma}}(\tau,\vec{\sigma})$, from Eq.(3.2) we get $\alpha^i(\tau,\vec{\sigma}) = \alpha^i(0,\vec{g}_{\vec{\Sigma}}(\tau,\vec{\sigma}))$. While on Σ_τ with $\tau > 0$ the position of the flux lines is identified by the Eulerian coordinates $\vec{\sigma} = \vec{\Sigma}(\tau,\vec{\sigma}_o)$ in the Eulerian point of view, in the Lagrangian one this position is identified by the Lagrangian coordinates $\alpha^i(\tau,\vec{\sigma}) = \alpha^i(0,\vec{g}_{\vec{\Sigma}}(\tau,\vec{\sigma}))$ [see also Eq.(3.9)].

If we remember the definition (2.12), we have

$$\frac{d\alpha^i(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))}{d\tau} = \frac{\partial\alpha^i(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))}{\partial\tau} + [I(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))]^i{}_{\check{r}}\,\frac{\partial\Sigma^{\check{r}}(\tau,\vec{\sigma}_o)}{\partial\tau} = 0, \tag{3.3}$$

so that

$$\frac{\partial \Sigma^{\check r}(\tau,\vec\sigma_o)}{\partial\tau} = -[I^{-1}(\tau,\vec\Sigma(\tau,\vec\sigma_o))]^{\check r}_i \frac{\partial\alpha^i(\tau,\vec\Sigma(\tau,\vec\sigma_o))}{\partial\tau} = +\frac{J^{\check r}(\tau,\vec\Sigma(\tau,\vec\sigma_o))}{J^\tau(\tau,\vec\Sigma(\tau,\vec\sigma_o))}. \tag{3.4}$$

Since we have

$$[I(\tau,\vec\Sigma(\tau,\vec\sigma_o))]^i_{\check r}\frac{\partial\Sigma^{\check r}(\tau,\vec\sigma_o)}{\partial\sigma_o^{\check s}} = \frac{\partial\alpha^i_o(\vec\sigma_o)}{\partial\sigma_o^{\check s}}, \tag{3.5}$$

then we get

$$J^\tau(\tau,\vec\Sigma(\tau,\vec\sigma_o)) = -\det\left(I(\tau,\vec\Sigma(\tau,\vec\sigma_o))\right) = -\det{}^{-1}\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\det\left(\frac{\partial\alpha_o(\vec\sigma_o)}{\partial\sigma_o}\right) =$$

$$= n_o(\vec\sigma_o)\det{}^{-1}\left(\frac{\partial\Sigma}{\partial\sigma_o}\right). \tag{3.6}$$

The function

$$n_o(\vec\sigma_o) = -\det\left(\frac{\partial\alpha_o(\vec\sigma_o)}{\partial\sigma_o}\right), \tag{3.7}$$

is *the particle numerical density on the Cauchy hyper-surface* $\Sigma_{\tau=0}$ *and is known from the fluid initial conditions*. With the previous results, we can write

$$J^\tau(\tau,\vec\Sigma(\tau,\vec\sigma_o)) = n_o(\vec\sigma_o)\det{}^{-1}\left(\frac{\partial\Sigma}{\partial\sigma_o}\right),$$

$$J^{\check r}(\tau,\vec\Sigma(\tau,\vec\sigma_o)) = n_o(\vec\sigma_o)\det{}^{-1}\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\frac{\partial\Sigma^{\check r}}{\partial\tau}. \tag{3.8}$$

Let us notice that, as said in the Introduction, any functional of the thermodynamical functions admits many expressions $\mathcal{G}(\tau,\vec\sigma) = \tilde{\mathcal{G}}(z(\tau,\vec\sigma)) = \mathcal{G}(\tau,\vec\Sigma(\tau,\vec\sigma_o)) = \hat{\mathcal{G}}(\tau,\vec\sigma_o) = \mathcal{G}_\alpha(\tau,\vec\alpha(\tau,\vec\sigma))$ (we have added the expression in terms of the Lagrangian comoving coordinates) [6]. In particular $\widehat{\mathcal{G}}(\tau,\vec\sigma_o)$ and $\mathcal{G}(\tau,\vec\sigma)$ are the expressions in the Lagrangian (or material) and Eulerian (or local) point of view respectively. While $\hat{\mathcal{G}}(\tau,\vec\sigma_o)$ is defined only on $\Sigma_{\tau=0}$, $\mathcal{G}(\tau,\vec\sigma)$ gives the expression on arbitrary hyper-surfaces $\Sigma_{\tau\neq 0}$. We have $\mathcal{G}(\tau,\vec\sigma)|_{\vec\sigma=\vec\Sigma(\tau,\vec\sigma_o)} = \hat{\mathcal{G}}(\tau,\vec\sigma_o)$ and $\mathcal{G}(\tau,\vec\sigma) = \hat{\mathcal{G}}(\tau,\vec\sigma_o)|_{\vec\sigma_o=\vec g_\Sigma(\tau,\vec\sigma)}$. Therefore we get the following equation connecting them

[6] Note that in general each of these functions is actually a functional of the associated coordinates and of their time and spatial gradients.

$$\mathcal{G}(\tau,\vec{\sigma}) = \int d^3\sigma_o\, \det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\delta^3(\vec{\sigma}-\vec{\Sigma}(\tau,\vec{\sigma}_o))\hat{\mathcal{G}}(\tau,\vec{\sigma}_o). \tag{3.9}$$

The particle density $\hat{n}$ is still given by Eq.(2.18) with the $J^{\check{A}}$ given by Eqs. (3.8). Then, introducing the notation

$$\mathcal{R}(\tau,\vec{\sigma}_o) = \sqrt{g_{\tau\tau}(\tau,\vec{\Sigma}) + 2\, g_{\tau\check{r}}(\tau,\vec{\Sigma})\,\frac{\partial\Sigma^{\check{r}}}{\partial\tau} + g_{\check{r}\check{s}}(\tau,\vec{\Sigma})\,\frac{\partial\Sigma^{\check{r}}}{\partial\tau}\,\frac{\partial\Sigma^{\check{s}}}{\partial\tau}}\,|_{\vec{\Sigma}=\vec{\Sigma}(\tau,\vec{\sigma}_o)}, \tag{3.10}$$

we get

$$\hat{n}(\tau,\vec{\sigma}_o) = n(\tau,\vec{\sigma})|_{\vec{\sigma}=\vec{\Sigma}(\tau,\vec{\sigma}_o)} = n_o(\vec{\sigma}_o)\,\frac{\det^{-1}\left(\frac{\partial\Sigma(\tau,\vec{\sigma}_o)}{\partial\sigma_o}\right)}{\sqrt{g(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))}}\,\mathcal{R}(\tau,\vec{\sigma}_o). \tag{3.11}$$

Instead the entropy per particle s can be rewritten as a function dependent on $\vec{\sigma}_o$ alone

$$s(\alpha(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))) = s(\alpha_o(\vec{\sigma}_o)) = s_o(\vec{\sigma}_o), \tag{3.12}$$

and it is given with the initial conditions. In this case the constraint (2.9) is trivially satisfied being equivalent to

$$\frac{\partial}{\partial\tau}\, s_o(\vec{\sigma}_o) = 0. \tag{3.13}$$

Analogously the particle number constraint is satisfied by construction being in particular

$$\mathcal{N} = \int_{V_\alpha(\tau)} d^3\sigma\, J^\tau(\tau,\vec{\sigma}) = \int_{V_\alpha(0)} d^3\sigma_o\, n_o(\vec{\sigma}_o). \tag{3.14}$$

The action in the new coordinates is

$$\begin{aligned} S &= -\int d\tau \int_{\Sigma_\tau} d^3\sigma\,\sqrt{g(\tau,\vec{\sigma})} \\ &\qquad \int_{\Sigma_{\tau=0}} d^3\sigma_o\,\delta^3(\vec{\sigma}-\vec{\Sigma}(\tau,\vec{\sigma}_o))\,\det\left(\frac{\partial\Sigma(\tau,\vec{\sigma}_o)}{\partial\sigma_o}\right)\,\rho(\hat{n}(\tau,\vec{\sigma}_o), s_o(\vec{\sigma}_o)\,) = \\ &= -\int_{\Sigma_{\tau=0}} d\tau\, d^3\sigma_o\,\sqrt{g(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))}\,\det\left(\frac{\partial\Sigma(\tau,\vec{\sigma}_o)}{\partial\sigma_o}\right)\,\rho(\hat{n}(\tau,\vec{\sigma}_o), s_o(\vec{\sigma}_o)), \end{aligned} \tag{3.15}$$

with the spatial integral restricted to the volume $V_\alpha(0)$ in $\Sigma_{\tau=0}$.

The equations of motion of the fluid variables derive from the stationarity of the new action with respect to the variations $\delta\vec{\Sigma}(\tau,\vec{\sigma}_o)$. If the four-velocity $\widehat{U}^{\check{A}}(\tau,\vec{\sigma}_o)$ is defined by Eqs. (2.10) with the new expression for the $J^{\check{A}}(\tau,\vec{\sigma}_o)$ given by Eqs. (3.8), that is

$$\widehat{U}^{\check{A}}(\tau,\vec{\sigma}_o) = \frac{1}{\mathcal{R}(\tau,\vec{\sigma}_o)} \frac{\partial \Sigma^{\check{A}}(\tau,\vec{\sigma}_o)}{\partial \tau}, \quad where \quad \Sigma^{\check{A}} = (\tau, \Sigma^{\check{r}}), \tag{3.16}$$

then the equations of motion are

$$-\widehat{U}^{\tau}(\tau,\vec{\sigma}_o) \frac{\partial \Sigma^{\check{r}}(\tau,\vec{\sigma}_o)}{\partial \sigma_o^{\check{s}}} \frac{\partial}{\partial \tau} \widehat{V}_{\check{r}}(\tau,\vec{\sigma}_o) + \widehat{U}^{\check{A}}(\tau,\vec{\sigma}_o) \frac{\partial}{\partial \sigma_o^{\check{s}}} \widehat{V}_{\check{A}}(\tau,\vec{\sigma}_o) - \widehat{T}(\tau,\vec{\sigma}_o) \frac{\partial s_o(\vec{\sigma}_o)}{\partial \sigma_o^{\check{s}}} = 0, \tag{3.17}$$

with $\widehat{V}^{\check{A}}(\tau,\vec{\sigma}_o) = \widehat{\mu}(\tau,\vec{\sigma}_o)\, \widehat{U}^{\check{A}}(\tau,\vec{\sigma}_o)$. Let us prove that these equations are equivalent to Eqs. (2.19). First we can observe that from Eq.(3.9) we get

$$\frac{\partial}{\partial \sigma^{\check{r}}} \mathcal{G}(\tau,\vec{\sigma}) = \int d^3\sigma_o \, \det\left(\frac{\partial \Sigma(\tau,\vec{\sigma}_o)}{\partial \sigma_o}\right) \delta^3(\vec{\sigma} - \vec{\Sigma}(\tau,\vec{\sigma}_o))\, K_{\check{r}}^{\check{s}}(\tau,\vec{\sigma}_o) \frac{\partial}{\partial \sigma_o^{\check{s}}} \hat{\mathcal{G}}(\tau,\vec{\sigma}_o),$$

$$\frac{\partial}{\partial \tau} \mathcal{G}(\tau,\vec{\sigma}) = \int d^3\sigma_o \, \det\left(\frac{\partial \Sigma(\tau,\vec{\sigma}_o)}{\partial \sigma_o}\right) \delta^3(\vec{\sigma} - \vec{\Sigma}(\tau,\vec{\sigma}_o)) \times$$

$$\times \left(\frac{\partial}{\partial \tau} - \frac{\partial \Sigma^{\check{r}}(\tau,\vec{\sigma}_o)}{\partial \tau} K_{\check{r}}^{\check{s}}(\tau,\vec{\sigma}_o) \frac{\partial}{\partial \sigma_o^{\check{s}}}\right) \hat{\mathcal{G}}(\tau,\vec{\sigma}_o), \tag{3.18}$$

where we have adopted the notation

$$K_{\check{r}}^{\check{s}}(\tau,\vec{\sigma}_o) \frac{\partial \Sigma^{\check{u}}(\tau,\vec{\sigma}_o)}{\partial \sigma_o^{\check{s}}} = \delta_{\check{r}}^{\check{u}}. \tag{3.19}$$

Using the rules (3.18) on the equations of motion (2.19) we can verify that Eqs. (3.17) are the material (Lagrangian) representation of the local (Eulerian) equations (2.19). [7] Therefore the action (3.15) defines the correct fluid equations of motion in the Lagrangian (material) point of view.

For the stress-energy tensor we get

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[7] Let us observe that we get

$$\frac{D}{D\tau} \mathcal{G}(\tau,\vec{\sigma}) = \frac{\partial \mathcal{G}(\tau,\vec{\sigma})}{\partial \tau} - \vec{v}(\tau,\vec{\sigma}) \cdot \frac{\partial}{\partial \vec{\sigma}} \mathcal{G}(\tau,\vec{\sigma}) =$$

$$= \int_{\Sigma_{\tau=0}} d^3\sigma_o \, \det\Big(\frac{\partial \Sigma(\tau,\vec{\sigma}_o)}{\partial \sigma_o}\Big) \delta^3(\vec{\sigma} - \vec{\Sigma}(\tau,\vec{\sigma}_o)) \frac{\partial \hat{\mathcal{G}}(\tau,\vec{\sigma}_o)}{\partial \tau},$$

with $\vec{v}(\tau,\vec{\sigma}) = \Big(\frac{\partial \vec{\Sigma}}{\partial \tau}(\tau,\vec{\sigma}_o)\Big)|_{\vec{\sigma}_o = \vec{g}_\Sigma(\tau,\vec{\sigma})}$ and with $\frac{D}{D\tau}$ denoting the *material temporal derivative.*

$$T^{\check{A}\check{B}}(\tau,\vec{\sigma}) = -\frac{2}{\sqrt{g(\tau,\vec{\sigma})}}\,\frac{\delta S}{\delta g_{\check{A}\check{B}}(\tau,\vec{\sigma})} =$$

$$= \int d^3\sigma_o\, \det\left(\frac{\partial\Sigma(\tau,\vec{\sigma}_o)}{\partial\sigma_o}\right)\, \delta^3(\vec{\sigma} - \vec{\Sigma}(\tau,\vec{\sigma}_o))\, \widehat{T}^{\check{A}\check{B}}(\tau,\vec{\sigma}_o), \tag{3.20}$$

where

$$\widehat{T}^{\check{A}\check{B}}(\tau,\vec{\sigma}_o) = (\widehat{\rho}(\tau,\vec{\sigma}_o) + \widehat{p}(\tau,\vec{\sigma}_o))\; \widehat{U}^{\check{A}}(\tau,\vec{\sigma}_o)\, \widehat{U}^{\check{B}}(\tau,\vec{\sigma}_o) - \widehat{p}(\tau,\vec{\sigma}_o)\, g^{\check{A}\check{B}}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o)). \tag{3.21}$$

Obviously Eqs.(3.17), being equivalent to Eqs.(2.19), imply the conservation of the stress-energy tensor (3.20). Again there are no equations of motion for the $z^\mu(\tau,\vec{\sigma})$'s, so that they remain gauge variables also for the action (3.15).

IV. THE HAMILTONIAN FORMULATION.

Let us study the Hamiltonian formulation of relativistic perfect fluid dynamics implied by the action (3.15). In this Section we use the notation $f(\Sigma)$ instead of $f(\tau, \vec{\Sigma}(\tau, \vec{\sigma}_o))$ for the sake of simplicity. The action (3.15) depends on the generalized Eulerian coordinates $\vec{\Sigma}(\tau, \vec{\sigma})$ of the fluid and on the embeddings $z^\mu(\tau, \vec{\sigma})$ as configurational variables.

A. The First Class Constraints.

We can define the following canonical momenta: the *momentum density for the fluid*

$$K_{\check{r}}(\tau, \vec{\sigma}_o) = -\frac{\delta S}{\delta\left(\frac{\partial \Sigma^{\check{r}}(\tau, \vec{\sigma}_o)}{\partial \tau}\right)} = n_o(\vec{\sigma}_o)\, \frac{\partial \rho}{\partial \hat{n}} \left[\frac{g_{\check{r}\tau}(\Sigma) + g_{\check{r}\check{s}}(\Sigma)\frac{\partial \Sigma^{\check{s}}}{\partial \tau}}{\mathcal{R}}(\tau, \vec{\sigma}_o)\right], \tag{4.1}$$

and the *momentum density of the embedding*

$$\rho_\mu(\tau, \vec{\sigma}) = -\frac{\delta S}{\delta z^\mu_\tau(\tau, \vec{\sigma})} =$$

$$= \int d^3\sigma_o\, \det\left(\frac{\partial \Sigma(\tau, \vec{\sigma}_o)}{\partial \sigma_o}\right) \delta^3(\vec{\sigma} - \vec{\Sigma}(\tau, \vec{\sigma}_o))\, \left(\rho(\hat{n}, s) - \frac{\partial \rho}{\partial \hat{n}} \hat{n}\right) \frac{\partial \sqrt{g(\Sigma)}}{\partial z^\mu_\tau(\Sigma)} +$$

$$+ \int d^3\sigma_o\, \delta(\vec{\sigma} - \vec{\Sigma}(\tau, \vec{\sigma}_o))\, n_o(\vec{\sigma}_o)\, \frac{\partial \rho}{\partial \hat{n}}\, \frac{z_{\mu\tau}(\Sigma) + z_{\mu\check{r}}(\Sigma)\frac{\partial \Sigma^{\check{r}}}{\partial \tau}}{\mathcal{R}}(\tau, \vec{\sigma}_o). \tag{4.2}$$

The Darboux canonical basis of phase space is represented in the following table

$$\begin{array}{|c|c|} \hline z^\mu(\tau, \vec{\sigma}) & \Sigma^{\check{r}}(\tau, \vec{\sigma}_o) \\ \rho_\mu(\tau, \vec{\sigma}) & K_{\check{r}}(\tau, \vec{\sigma}_o) \\ \hline \end{array} \tag{4.3}$$

and we assume the following non null Poisson Brackets

$$\{z^\mu(\tau, \vec{\sigma}), \rho_\nu(\tau, \vec{\sigma}^{\prime})\} = -\eta^\mu_\nu\, \delta^3(\vec{\sigma} - \vec{\sigma}^{\prime}),$$

$$\{\Sigma^{\check{r}}(\tau, \vec{\sigma}_o), K_{\check{s}}(\tau, \vec{\sigma}^{\prime}_o)\} = -\delta^{\check{r}}_{\check{s}}\, \delta^3(\vec{\sigma}_o - \vec{\sigma}^{\prime}_o). \tag{4.4}$$

As we said at the end of Sections II and III, we expect that the momentum density (4.2) is equivalent to four first class constraints. Since we have

$$\frac{\partial\sqrt{g(\Sigma)}}{\partial z^\mu_\tau(\Sigma)}\, z^\mu_{\check r}(\Sigma)=0,\qquad \frac{\partial\sqrt{g(\Sigma)}}{\partial z^\mu_\tau(\Sigma)}\, l^\mu(\Sigma)=\sqrt{\gamma(\Sigma)}, \tag{4.5}$$

then, using Eq.(4.1), Eq.(4.2) implies

$$\rho_\mu(\tau,\vec\sigma)z^\mu_{\check r}(\tau,\vec\sigma)=\int d^3\sigma_o\,\delta^3(\vec\sigma-\vec\Sigma(\tau,\vec\sigma_o))K_{\check r}(\tau,\vec\sigma_o),$$

$$\begin{aligned}\rho_\mu(\tau,\vec\sigma)l^\mu(\tau,\vec\sigma)&=\int d^3\sigma_o\,\delta^3(\vec\sigma-\vec\Sigma(\tau,\vec\sigma_o))\times\\ &\times\Big[\det\left(\frac{\partial\Sigma(\tau,\vec\sigma_o)}{\partial\sigma_o}\right)\sqrt{\gamma(\Sigma)}\,\left(\rho(\hat n,s)-\frac{\partial\rho}{\partial\hat n}\hat n\right)+\\ &+\frac{n_o^2(\vec\sigma_o)}{\sqrt{\gamma(\Sigma)}\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)}\,\frac{1}{\hat n}\,\frac{\partial\rho}{\partial\hat n}\Big](\tau,\vec\sigma_o).\end{aligned} \tag{4.6}$$

The first expression is already in a constraint form. In the right-hand side of the second one the only dependence on the velocities is in the density $\hat n(\tau,\vec\sigma_o)$ [see the definition (3.11)]. Nevertheless we observe that from the definition (4.1) we obtain

$$\begin{aligned}\gamma^{\check r\check s}(\Sigma)K_{\check r}(\tau,\vec\sigma_o)K_{\check s}(\tau,\vec\sigma_o)&=\left(n_o(\vec\sigma_o)\,\frac{\partial\rho}{\partial\hat n}(\hat n,s)\right)^2\\ &\left[-\frac{n_o^2(\vec\sigma_o)}{\hat n^2(\tau,\vec\sigma_o)}\left(\frac{1}{\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\sqrt{\gamma(\Sigma)}}\right)^2+1\right].\end{aligned} \tag{4.7}$$

Then we can replace $\hat n(\tau,\vec\sigma_o)$ with the (implicit) solution $X(\tau,\vec\sigma_o)$ of the equation

$$\begin{aligned}\gamma^{\check r\check s}(\Sigma)K_{\check r}(\tau,\vec\sigma_o)K_{\check s}(\tau,\vec\sigma_o)&=\left(n_o(\vec\sigma_o)\,\frac{\partial\rho}{\partial X}(X(\tau,\vec\sigma_o),s)\right)^2\\ &\left[-\frac{n_o^2(\vec\sigma_o)}{X^2(\tau,\vec\sigma_o)}\left(\frac{1}{\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\sqrt{\gamma(\Sigma)}}\right)^2+1\right].\end{aligned} \tag{4.8}$$

It is evident by inspection that $X(\tau,\vec\sigma_o)$ is a function of the canonical variables alone. In other words $X(\tau,\vec\sigma_o)$ is independent from the τ-derivative of the z^μ's and Σ's. It depends on the initial particle density $n_o(\vec\sigma_o)$, on the Eulerian coordinates through $\frac{n_o(\vec\sigma_o)}{\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\sqrt{\gamma(\Sigma)}}$ and on the Eulerian momenta through $\gamma^{\check r\check s}(\Sigma)K_{\check r}(\tau,\vec\sigma_o)K_{\check s}(\tau,\vec\sigma_o)$.

Then we have the following Dirac constraints:

$$\mathcal{H}_{\check r}(\tau,\vec\sigma) = \rho_\mu(\tau,\vec\sigma) z^\mu_{\check r}(\tau,\vec\sigma) - \int d^3\sigma_o \cdot \delta^3(\vec\sigma - \vec\Sigma(\tau,\vec\sigma_o)) K_{\check r}(\tau,\vec\sigma_o) \approx 0,$$

$$\mathcal{H}_\perp(\tau,\vec\sigma) = \rho_\mu(\tau,\vec\sigma) l^\mu(\tau,\vec\sigma) - \int d^3\sigma_o \cdot \delta^3(\vec\sigma - \vec\Sigma(\tau,\vec\sigma_o)) \times$$

$$\left[\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\sqrt{\gamma(\Sigma)}\left(\rho(X,s) - \frac{\partial\rho}{\partial X} X\right) + \frac{n_o^2(\vec\sigma_o)}{\sqrt{\gamma(\Sigma)}\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)}\frac{1}{X}\frac{\partial\rho}{\partial X}\right](\tau,\vec\sigma_o) \approx 0. \tag{4.9}$$

As shown in Appendix B, following Ref. [6], only in few cases, including the dust, the photon gas and a polytropic with $n = \frac{1}{2}$ or $\gamma = 1 + \frac{1}{n} = 3$, a closed form of Eq.(4.9) can be obtained.

Working on the implicit definition of $X(\tau,\vec\sigma_o)$ given by Eq.(4.8) and using the results of Appendix C, it is possible to prove that the previous constraints satisfy the Dirac algebra

$$\{\mathcal{H}_{\check r}(\tau,\vec\sigma), \mathcal{H}_{\check s}(\tau,\vec\sigma')\} = \mathcal{H}_{\check r}(\tau,\vec\sigma')\frac{\partial}{\partial\sigma'^{\check s}}\delta(\vec\sigma - \vec\sigma') - \mathcal{H}_{\check s}(\tau,\vec\sigma)\frac{\partial}{\partial\sigma^{\check r}}\delta^3(\vec\sigma - \vec\sigma'),$$

$$\{\mathcal{H}_\perp(\tau,\vec\sigma), \mathcal{H}_\perp(\tau,\vec\sigma')\} = \Big[\mathcal{H}_{\check r}(\tau,\vec\sigma)\gamma^{\check r\check s}(\tau,\vec\sigma) + \mathcal{H}_{\check r}(\tau,\vec\sigma')\gamma^{\check r\check s}(\tau,\vec\sigma')\Big]\frac{\partial}{\partial\sigma^{\check s}}\delta^3(\vec\sigma - \vec\sigma'),$$

$$\{\mathcal{H}_\perp(\tau,\vec\sigma), \mathcal{H}_{\check r}(\tau,\vec\sigma')\} = \mathcal{H}_\perp(\tau,\vec\sigma')\frac{\partial}{\partial\sigma'^{\check r}}\delta^3(\vec\sigma - \vec\sigma'), \tag{4.10}$$

so that they are first class constraint.

It is convenient to rewrite the constraints in the form

$$\mathcal{H}^\mu(\tau,\vec\sigma) = \mathcal{H}_\perp(\tau,\vec\sigma) l^\mu(\tau,\vec\sigma) + \mathcal{H}_{\check r}(\tau,\vec\sigma)\gamma^{\check r\check s}(\tau,\vec\sigma) z^\mu_{\check s}(\tau,\vec\sigma) =$$

$$= \rho^\mu(\tau,\vec\sigma) - l^\mu(\tau,\vec\sigma)\int d^3\sigma_o \cdot \delta^3(\vec\sigma - \vec\Sigma(\tau,\vec\sigma_o)) \times$$

$$\left[\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\sqrt{\gamma(\Sigma)}\left(\rho(X,s) - \frac{\partial\rho}{\partial X} X\right) + \frac{n_o^2(\vec\sigma_o)}{\sqrt{\gamma(\Sigma)}\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)}\frac{1}{X}\frac{\partial\rho}{\partial X}\right](\tau,\vec\sigma_o) -$$

$$- z^\mu_{\check r}(\tau,\vec\sigma)\int d^3\sigma_o \cdot \delta^3(\vec\sigma - \vec\Sigma(\tau,\vec\sigma_o)) K_{\check r}(\tau,\vec\sigma_o) \approx 0, \tag{4.11}$$

because then we get

$$\{\mathcal{H}^\mu(\tau,\vec{\sigma}), \mathcal{H}^\nu(\tau,\vec{\sigma}\,')\} = 0. \tag{4.12}$$

The Hamiltonian gauge transformations generated by these constraints change the form and the coordinatization of the space-like hyper-surfaces Σ_τ. Therefore the embeddings $z^\mu(\tau,\vec{\sigma})$ are the *gauge variables* of this special relativistic general covariance according to which the description of isolated systems (here the perfect fluid) does not depend from the choice of the 3+1 splitting of Minkowski space-time.

The Dirac Hamiltonian is $H_D = \int d^3\sigma\, \lambda_\mu(\tau,\vec{\sigma})\, \mathcal{H}^\mu(\tau,\vec{\sigma})$, where the λ_μ's are arbitrary Dirac multipliers. Since only the embedding carries Lorentz indices, the generators of the Poincare' group are $p_\mu = \int d^3\sigma\, \rho_\mu(\tau,\vec{\sigma})$ and $J^{\mu\nu} = \int d^3\sigma\, \Big(z^\mu\, \rho^\nu - z^\nu\, \rho^\mu\Big)(\tau,\vec{\sigma})$.

B. The Restriction to Space-Like Hyper-Planes.

Following Refs. [8,10,3] let us restrict ourselves to foliations whose leaves are space-like hyper-planes by adding the gauge-fixings:

$$\zeta^\mu(\tau,\vec{\sigma}) = z^\mu(\tau,\vec{\sigma}) - x^\mu(\tau) - b^\mu_{\check{r}}(\tau)\sigma^{\check{r}} \approx 0 \tag{4.13}$$

With this condition many geometrical quantities take a trivial expression, in particular we have:

$$\begin{aligned}
&z^\mu_{\check{r}}(\tau,\vec{\sigma}) \approx b^\mu_{\check{r}}(\tau),\\
&z^\mu_{\check{r}}(\tau,\vec{\sigma}) \approx \dot{x}^\mu(\tau) + \dot{b}^\mu_{\check{r}}(\tau)\sigma^{\check{r}},\\
&g_{\check{r}\check{s}}(\tau,\vec{\sigma}) \approx -\delta_{\check{r}\check{s}},\;\; \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma}) \approx -\delta^{\check{r}\check{s}},\;\; \gamma(\tau,\vec{\sigma}) \approx 1.
\end{aligned} \tag{4.14}$$

We also introduce the following notation for the unit normal to the hyper-planes defined in Eq.(A3)

$$b^\mu_\tau(\tau) = l^\mu(\tau) \approx l^\mu(\tau,\vec{\sigma}) \tag{4.15}$$

The hyper-planes define a *true global foliation* only if the normal l_μ is τ-independent, because only in this case the hyper-planes are parallel and not intersecting and there is a one to one global correspondences between points z^μ and coordinates $\tau, \vec{\sigma}$. We ignore from now on this observation, but we shall return on the consequences of the $l^\mu = constant$ request at the end of this Section.

Since we have

$$\{\zeta^{\mu}(\tau,\vec{\sigma}),\mathcal{H}_{\nu}(\tau,\vec{\sigma}')\} = -\eta^{\mu}_{\nu}\delta^{3}(\vec{\sigma}-\vec{\sigma}'), \tag{4.16}$$

it is possible to define the following Dirac brackets

$$\{A,B\}^{*} = \{A,B\} - \int d^{3}\sigma \cdot [\{A,\zeta^{\mu}(\tau,\vec{\sigma})\}\{\mathcal{H}_{\mu}(\tau,\vec{\sigma}),B\} - \{A,\mathcal{H}^{\mu}(\tau,\vec{\sigma})\}\{\zeta_{\mu}(\tau,\vec{\sigma}),B\}]\,. \tag{4.17}$$

In the reduced phase space the hyper-surface canonical variables $z^{\mu}(\tau,\vec{\sigma})$, $\rho_{\mu}(\tau,\vec{\sigma})$ are reduced to only ten canonical pairs and the first class constraints (4.11) are reduced to only ten [8,6].

We can verify that four pairs of canonical variables are the centroid $x^{\mu}(\tau)$, used as origin of the 3-coordinates on the hyper-planes, and the conjugate momentum $p^{\mu}(\tau)$

$$\{x^{\mu}(\tau),p^{\nu}(\tau)\}^{*} = -\eta^{\mu\nu}. \tag{4.18}$$

Since p^{μ} is the Poincare' generator of the translations, it describes the total 4-momentum of the system. As a consequence we have the following decomposition for the canonical generators of Lorentz transformations inside the Poincare' algebra

$$J^{\mu\nu}(\tau) = x^{\mu}(\tau)p^{\nu}(\tau) - x^{\nu}(\tau)p^{\mu}(\tau) + S^{\mu\nu}(\tau). \tag{4.19}$$

The remaining canonical variables defining the hyper-planes are the variables $\phi_{\lambda}(\tau)$ ($\lambda = 1,...,6$) that parametrize the orthonormal tetrad $b^{\mu}_{\check{A}}(\tau)$ such that

$$b^{\mu}_{\check{A}}(\tau)\eta_{\mu\nu}b^{\nu}_{\check{B}}(\tau) = \eta_{\check{A}\check{B}}, \tag{4.20}$$

and the associated conjugate variables $T_{\lambda}(\tau)$. Nevertheless it is possible [8] to use a set of redundant, non independent and non canonical variables. These are the tetrads $b^{\mu}_{\check{A}}(\tau)$ and the spin tensor $S^{\mu\nu}(\tau)$ of Eq.(4.19), if they satisfy the following Poisson brackets [8]

$$\begin{aligned}\{S^{\mu\nu}(\tau),b^{\rho}_{\check{A}}(\tau)\}^{*} &= \eta^{\rho\nu}b^{\mu}_{\check{A}}(\tau) - \eta^{\rho\mu}b^{\nu}_{\check{A}}(\tau),\\ \{S^{\mu\nu}(\tau),S^{\rho\sigma}(\tau)\}^{*} &= C^{\mu\nu\rho\sigma}_{\alpha\beta}S^{\alpha\beta}(\tau),\end{aligned} \tag{4.21}$$

[8]They are the Dirac brackets enforcing the fulfillment [24] of Eqs.(4.20).

with the Lorentz group constant structures:

$$C^{\mu\nu\rho\sigma}_{\alpha\beta} = \eta^{\nu}_{\alpha}\eta^{\rho}_{\beta}\eta^{\mu\sigma} - \eta^{\mu}_{\alpha}\eta^{\rho}_{\beta}\eta^{\nu\sigma} - \eta^{\nu}_{\alpha}\eta^{\sigma}_{\beta}\eta^{\mu\rho} + \eta^{\mu}_{\alpha}\eta^{\sigma}_{\beta}\eta^{\nu\rho}. \tag{4.22}$$

Finally we have the unchanged fluids variables:

$$\{\Sigma^{\check r}(\tau,\vec\sigma_o), K^{\check s}(\tau,\vec\sigma'_o)\}^* = \delta(\vec\sigma_o - \vec\sigma'_o)\,\delta^{\check r\check s}. \tag{4.23}$$

In conclusion this non-Darboux canonical basis of the reduced phase space can be represented in the table:

$$\begin{array}{|cc|c|} \hline x^\mu(\tau) & S^{\mu\nu}(\tau) & \Sigma^{\check r}(\tau,\vec\sigma_o) \\ p^\mu(\tau) & b^\mu_{\check A}(\tau) & K^{\check r}(\tau,\vec\sigma_o) \\ \hline \end{array}. \tag{4.24}$$

The stability in time of the gauge fixings (4.13) force the arbitrary Dirac multiplier to take the reduced form:

$$\frac{\partial}{\partial\tau}\zeta(\tau,\vec\sigma) \approx 0 \Rightarrow \lambda^\mu(\tau,\vec\sigma) \approx \lambda^\mu(\tau) + \lambda^{\mu\nu}(\tau)\eta_{\nu\rho}b^\rho_{\check r}(\tau)\sigma^{\check r}. \tag{4.25}$$

The Dirac Hamiltonian become

$$H_D = \lambda^\mu(\tau)H_\mu(\tau) + \lambda^{\mu\nu}(\tau)H_{\mu\nu}(\tau), \tag{4.26}$$

where, if we define

$$\begin{aligned}
\mathcal{M}(\tau) &= \int d^3\sigma_o\left[\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\left(\rho(X,s) - \frac{\partial\rho}{\partial X}X\right) + \frac{n_o^2(\vec\sigma_o)}{\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)}\frac{1}{X}\frac{\partial\rho}{\partial X}\right](\tau,\vec\sigma_o),\\
\mathcal{P}^{\check r}(\tau) &= \int d^3\sigma_o\, K^{\check r}(\tau,\vec\sigma_o),\\
\mathcal{J}^{\tau\check r}(\tau) &= \mathcal{K}^{\check r}(\tau) = -\int d^3\sigma_o\,\Sigma^{\check r}(\tau,\vec\sigma_o)\times\\
&\quad\times\left[\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\left(\rho(X,s) - \frac{\partial\rho}{\partial X}X\right) + \frac{n_o^2(\vec\sigma_o)}{\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)}\frac{1}{X}\frac{\partial\rho}{\partial X}\right](\tau,\vec\sigma_o),\\
\mathcal{J}^{\check r\check s}(\tau) &= \int d^3\sigma_o\left[\Sigma^{\check r}(\tau,\vec\sigma_o)K^{\check s}(\tau,\vec\sigma_o) - \Sigma^{\check s}(\tau,\vec\sigma_o)K^{\check r}(\tau,\vec\sigma_o)\right],
\end{aligned} \tag{4.27}$$

we have the following form of the ten remaining first class constraints

$$H^\mu(\tau) = p^\mu(\tau) - l^\mu(\tau)\mathcal{M}(\tau) + b^\mu_{\check r}(\tau)\mathcal{P}^{\check r}(\tau) \approx 0,$$

$$H^{\mu\nu}(\tau) = S^{\mu\nu}(\tau) + [b^\mu_{\check r}(\tau)l^\nu(\tau) - b^\nu_{\check r}(\tau)l^\mu(\tau)]\,\mathcal{J}^{\tau\check r}(\tau) +$$

$$- \frac{1}{2}\,[b^\mu_{\check r}(\tau)b^\nu_{\check s}(\tau) - b^\mu_{\check s}(\tau)b^\mu_{\check r}(\tau)]\,\mathcal{J}^{\check r\check s}(\tau) \approx 0. \qquad (4.28)$$

They satisfy the Poisson algebra

$$\{H^\mu(\tau), H^\nu(\tau)\} = \{H^\mu(\tau), H^{\alpha\beta}(\tau)\} = 0,$$

$$\{H^{\mu\nu}(\tau), H^{\rho\sigma}(\tau)\} = C^{\mu\nu\rho\sigma}_{\alpha\beta}\, H^{\alpha\beta}(\tau) \approx 0. \qquad (4.29)$$

These ten first class constraints imply that $x^\mu(\tau)$, $b^\mu_{\check r}(\tau)$ are *gauge variables* and that the description is independent from the choice of the space-like hyper-planes.

We can discuss now the condition $l_\mu = constant$.

By using the brackets (4.21), the Hamiltonian (4.20) implies the following equations of motion for the normal $l_\mu(\tau)$

$$\frac{d}{d\tau} l_\mu(\tau) = 2\lambda_{\mu\nu}(\tau) l^\nu(\tau). \qquad (4.30)$$

Then the condition $l_\mu = constant$ restricts the arbitrariness of the $\lambda_{\mu\nu}(\tau)$'s with the condition

$$\lambda_{\mu\nu}(\tau) l^\nu = 0, \quad \Rightarrow \lambda^{\mu\nu}(\tau) = b^\mu_{\check r}(\tau)\, b^\nu_{\check s}(\tau)\, \lambda^{\check r\check s}(\tau), \quad \lambda^{\check r\check s}(\tau) = -\lambda^{\check s\check r}(\tau). \qquad (4.31)$$

In other words the condition $l_\mu = constant$ have to be interpreted as 3 gauge fixing conditions for the Lorentz transformations generated by the constraints $H^{\mu\nu}(\tau) \approx 0$, leaving only the possibility of Hamiltonian gauge rotations generated by $H_{\check r\check s}(\tau) = b^\mu_{\check r}(\tau)\, b^\nu_{\check s}(\tau)\, H_{\mu\nu}(\tau)$. This implies that a foliation with space-like hyper-planes has 7 gauge variables: $x^\mu(\tau)$ and 3 angles inside $b^\mu_{\check r}(\tau)$ describing the linear acceleration of the origin and the arbitrary rotation of the spatial axes with respect to an inertial frame, respectively. Correspondingly only 7 of the first class constraints (4.28) are independent, namely $H^\mu(\tau)$ and $H_{\check r\check s}(\tau)$.

V. THE REST FRAME INSTANT FORM.

In this Section, following Ref. [8], we define a new instant form of dynamics [24] called the *rest frame instant form*. This is done by selecting all the configurations of the isolated system with time-like conserved total 4-momentum and, for each of them, by choosing the foliation whose space-like hyper-planes are orthogonal to this 4-momentum. Physically these hyper-planes, named *Wigner hyper-planes*, correspond to the intrinsic rest frame of the configuration of the isolated system. Moreover we have to fix the four acceleration degrees of freedom of the centroid $x^\mu(\tau)$, reducing it to the world-line of an inertial observer.

Before doing this we have to recall the notion of *standard Wigner boost* and *Wigner rotation*. It is known that in the rest frame a timelike four-vector p^μ assume the standard form $\bar{p}^\mu = (\sqrt{p^2}, \vec{0})$. In the general theory of the induced representation of the Poincaré group a *standard Wigner boost* $L^\mu{}_\nu(p, \bar{p})$ for time-like Poincare' orbits is defined such that

$$L^\mu{}_\nu(p, \bar{p})\, \bar{p}^\nu = p^\mu. \tag{5.1}$$

We can use the rows of the *standard Wigner boost* matrix to define a orthonormal tetrad (we define $n^\mu = p^\mu/\sqrt{p^2}$)

$$\begin{aligned}
\epsilon^\mu_o(p) &= L^\mu{}_o(p, \bar{p}) = n^\mu, \\
\epsilon^\mu_r(p) &= L^\mu{}_r(p, \bar{p}) = \Big(- n_r; \delta^i_r - n^i n_r (1 + n_o)^{1/2} \Big).
\end{aligned} \tag{5.2}$$

We use the notation

$$\epsilon^\mu_A(p) = (\epsilon^\mu_o(p), \epsilon^\mu_r(p)). \tag{5.3}$$

The inverse of standard boost defines naturally the inverse tetrad

$$\epsilon^A_\mu(p) = L_\mu{}^A(p, \bar{p}), \tag{5.4}$$

such that

$$\begin{aligned}
\epsilon^o_\mu(p) &= n_\mu = \frac{p_\mu}{\sqrt{p^2}}, \\
\epsilon^r_\mu(p) &= \Big(\delta^{rs} u_s; \delta^r_s - \delta^{rs} \delta_{jh} n^h n_s (1 + n_o)^{1/2} \Big),
\end{aligned} \tag{5.5}$$

and

$$\epsilon^A_\mu(p)\epsilon^\nu_A(p) = \eta^\nu_\mu, \qquad \epsilon^A_\mu(p)\epsilon^\mu_B(p) = \eta^A_B,$$

$$\eta^{\mu\nu} = n^\mu n^\nu - \sum_r \epsilon^\mu_r(p)\epsilon^\nu_r(p), \qquad \eta_{AB} = \epsilon^\mu_A(p)\eta_{\mu\nu}\epsilon^\nu_B(p). \tag{5.6}$$

Moreover we can verify that

$$p_\mu \epsilon^\mu_r(p) = p^\mu \epsilon^r_\mu(p) = 0,$$

$$p^\mu \frac{\partial}{\partial p^\mu}\epsilon^A_\nu(p) = p^\mu \frac{\partial}{\partial p^\mu}\epsilon^\nu_A(p) = 0. \tag{5.7}$$

The standard boost can be used to define the *Wigner rotation* $R(\Lambda, p)$ associated to any Lorentz transformation Λ

$$L^{-1}(p, \bar{p})\Lambda^{-1}L(\Lambda p, \bar{p}) = \begin{pmatrix} 1 & 0 \\ & \\ 0 & R(\Lambda, p) \end{pmatrix}. \tag{5.8}$$

Then we have the following property

$$\epsilon^\mu_r(\Lambda p) = \Lambda^\mu{}_\nu \epsilon^\nu_s(p) R^s{}_r(\Lambda, p). \tag{5.9}$$

We can use the tetrad (5.3) to define a gauge fixing for the $b^\mu_{\check{A}}(\tau)$. Before doing this, it is useful to replace the centroid $x^\mu(\tau)$ with the new one

$$q^\mu(\tau) = x^\mu(\tau) + \frac{1}{2}\epsilon^A_\rho(p)\eta_{AB}\frac{\partial \epsilon^B_\sigma(p)}{\partial p_\mu} S^{\rho\sigma}(\tau), \tag{5.10}$$

and the table (4.24) with the following set of non canonical variables

$$\begin{array}{|cc|c|} \hline q^\mu(\tau) & S^{\mu\nu}(\tau) & \Sigma^{\check{r}}(\tau, \vec{\sigma}_o) \\ p^\mu(\tau) & b^\mu_{\check{A}}(\tau) & K^{\check{r}}(\tau, \vec{\sigma}_o) \\ \hline \end{array}. \tag{5.11}$$

The $q^\mu(\tau), p^\mu(\tau)$ and $\Sigma^{\check{r}}(\tau, \vec{\sigma}_o), K^{\check{r}}(\tau, \vec{\sigma}_o)$ are still canonical

$$\begin{aligned} \{q^\mu(\tau), q^\nu(\tau)\} = 0, &\qquad \{q^\mu(\tau), p^\nu(\tau)\} = -\eta^{\mu\nu}, \\ \{q^\mu(\tau), \Sigma^{\check{r}}(\tau, \vec{\sigma}_o)\} = 0, &\qquad \{q^\mu(\tau), K^{\check{r}}(\tau, \vec{\sigma}_o)\} = 0, \end{aligned} \tag{5.12}$$

but for $S^{\mu\nu}(\tau), b^\mu_{\check{A}}(\tau)$ we have the following non canonical brackets

$$\{q^\mu(\tau), S^{\alpha\beta}(\tau)\} \neq 0, \qquad \{q^\mu(\tau), b^\alpha_{\check{A}}(\tau)\} \neq 0. \tag{5.13}$$

Eq.(5.10) also implies the following new decomposition for the canonical generators of Lorentz transformations

$$J^{\mu\nu}(\tau) = q^\mu(\tau)p^\nu(\tau) - q^\nu(\tau)p^\mu(\tau) + \Omega^{\mu\nu}(\tau), \tag{5.14}$$

where

$$\Omega^{\mu\nu}(\tau) = S^{\mu\nu} - \frac{1}{2}\epsilon^A_\alpha(p)\eta_{AB}\left(\frac{\partial\epsilon^B_\beta(p)}{\partial p_\mu}p^\nu - \frac{\partial\epsilon^B_\beta(p)}{\partial p_\nu}p^\mu\right)S^{\alpha\beta}. \tag{5.15}$$

Since we have

$$\{q^\mu, \Omega^{hk}\} = 0, \qquad \{q^\mu, \Omega^{ok}\} \neq 0, \tag{5.16}$$

we see that q^μ is not a true four-vector.

If we restrict ourselves to configurations with $p^\mu(\tau)$ time-like ($p^2(\tau) > 0$), there exists a family of space-like hyper-planes orthogonal to $p^\mu(\tau)$ and we can select this family with the gauge fixing [8]

$$T^\mu_A(\tau) = b^\mu_{\check{A}=A}(\tau) - \epsilon^\mu_A(p) \approx 0, \tag{5.17}$$

where the index $\check{r}$ is enforced to coincide with r with transformation property given by Eq.(5.9). These gauge fixings imply $\lambda_{\mu\nu}(\tau) \approx 0$ [9] and we have the Dirac Hamiltonian

$$H_D = \lambda^\mu(\tau)H_\mu(\tau), \tag{5.18}$$

where

$$H^\mu(\tau) = p^\mu - n^\mu\mathcal{M}(\tau) + \epsilon^\mu_r(p)\mathcal{P}^r(\tau) \approx 0, \tag{5.19}$$

are the remaining four first class Dirac constraints saying that $x^\mu(\tau)$ and terefore $q^\mu(\tau)$ are gauge variables. Since we have $\dot{x}^\mu(\tau) \circeq \{x^\mu(\tau), H_D\} = -\lambda^\mu(\tau)$, we see that the centroid has an arbitrary (gauge) acceleration described by the remaining Dirac multipliers. It is useful to observe that we can rewrite the constraints and the Dirac Hamiltonian in the form (we choose the positive sheet of the mass hyperboloid)

[9] So that Eqs.(4.31) are satisfied.

$$\mathcal{H}(\tau) = n^\mu H_\mu(\tau) = \sqrt{p^2} - \mathcal{M}(\tau) \approx 0,$$

$$\mathcal{P}^r(\tau) = \epsilon^r_\mu(p) H^\mu(\tau) \approx 0,$$

$$H_D = \lambda(\tau)\,\mathcal{H} - \vec{\lambda}(\tau)\cdot\vec{\mathcal{P}}, \tag{5.20}$$

where, recalling the definitions (4.27), the quantity

$$\mathcal{M}(\tau) = \int d^3\sigma_o\, \Big[\det\Big(\frac{\partial\Sigma}{\partial\sigma_o}\Big)\,\Big(\rho(X,s) - \frac{\partial\rho}{\partial X}\,X\Big) + \frac{n_o^2(\vec{\sigma}_o)}{\det\big(\frac{\partial\Sigma}{\partial\sigma_o}\big)}\,\frac{1}{X}\,\frac{\partial\rho}{\partial X}\Big](\tau,\vec{\sigma}_o) =$$

$$\stackrel{def}{=} \int d^3\sigma_o\,\Delta(\tau,\vec{\sigma}_o) \tag{5.21}$$

is the *invariant mass* and where

$$\vec{\mathcal{P}}(\tau) = \int d^3\sigma_o\,\vec{K}(\tau,\vec{\sigma}_o) \approx 0, \tag{5.22}$$

is a constraint on the total momentum of the fluid inside the Wigner hyperplane: it gives the *rest-frame condition*.

See Appendix B for the expression of the invariant mass $\mathcal{M}$ for the dust, the photon gas and the polytropic with $n = \frac{1}{2}$.

After this gauge fixing the non canonical variables $S^{\mu\nu}$ can be everywhere substituted with the expression implied for them by the constraints $H^{\mu\nu}(\tau) \approx 0$ of Eqs.(4.28). By defining

$$\mathcal{J}^{rs}(\tau) = \int d^3\sigma_o\, \left[\Sigma^r(\tau,\vec{\sigma}_o)K^s(\tau,\vec{\sigma}_o) - \Sigma^s(\tau,\vec{\sigma}_o)K^r(\tau,\vec{\sigma}_o)\right],$$

$$\mathcal{J}^{\tau r}(\tau) = \mathcal{K}^r = -\int d^3\sigma_o\,\Sigma^r(\tau,\vec{\sigma}_o)\,\times$$

$$\Big[\det\Big(\frac{\partial\Sigma}{\partial\sigma_o}\Big)\sqrt{\gamma(\Sigma)}\,\Big(\rho(X,s) - \frac{\partial\rho}{\partial X}\,X\Big) + \frac{n_o^2(\vec{\sigma}_o)}{\sqrt{\gamma(\Sigma)}\,\det\big(\frac{\partial\Sigma}{\partial\sigma_o}\big)}\,\frac{1}{X}\,\frac{\partial\rho}{\partial X}\Big](\tau,\vec{\sigma}_o) =$$

$$= -\int d^3\sigma_o\,\Sigma^r(\tau,\vec{\sigma}_o)\,\Delta(\tau,\vec{\sigma}_o), \tag{5.23}$$

we get

$$S^{\mu\nu}(\tau) \approx \epsilon^\mu_A(p)\epsilon^\nu_B(p)\mathcal{J}^{AB}(\tau). \tag{5.24}$$

After the elimination of the variables $b^\mu_{\breve{A}}$, $S^{\mu\nu}$ with Eqs. (4.28) and (5.17), the reduced phase space is spanned by the variables

$$\boxed{\begin{array}{c|c} q^\mu(\tau) & \Sigma^r(\tau,\vec{\sigma}_o) \\ p^\mu(\tau) & K^r(\tau,\vec{\sigma}_o) \end{array}}. \tag{5.25}$$

Its canonical structure is defined by the new Dirac brackets

$$\{A,B\}^{**} = \{A,B\}^* - \Big[\{A,H^{\mu\nu}\}^* D^A_{\mu\nu\rho}(p)\{T^\rho_A, B\}^* + \{A, T^\mu_A\}^* D^A_{\mu\nu\rho}(p)\{H^{\nu\rho}, B\}^*\Big], \tag{5.26}$$

where

$$D^A_{\mu\nu\rho}(p) = \frac{1}{4}[\eta_{\mu\rho}\epsilon^A_\nu(p) - \eta_{\nu\rho}\epsilon^A_\mu(p)]. \tag{5.27}$$

The variables (5.25) are canonical with respect to this bracket

$$\begin{aligned} \{\Sigma^r(\tau,\vec{\sigma}_o), K^s(\tau,\vec{\sigma}'_o)\}^{**} &= \delta^{rs}\,\delta(\vec{\sigma}_o - \vec{\sigma}'_o), \\ \{q^\mu, q^\nu\}^{**} &= 0, \\ \{q^\mu, p^\nu\}^{**} &= -\eta^{\mu\nu}, \\ \{q^\mu, \Sigma^r(\tau,\vec{\sigma}_o)\}^{**} &= 0, \\ \{q^\mu, K^s(\tau,\vec{\sigma}_o)\}^{**} &= 0. \end{aligned} \tag{5.28}$$

On the contrary, with the new brackets the old centroid x^μ is not canonical, because we have

$$\{x^\mu, \Sigma^r(\tau,\vec{\sigma}_o)\}^{**} \neq 0, \qquad \{x^\mu, K^r(\tau,\vec{\sigma}_o)\}^{**} \neq 0, \tag{5.29}$$

and this explains why we introduced the centroid $q^\mu(\tau)$.

After the gauge fixing we have

$$\Omega^{ij}(\tau) \approx \delta^{ir}\delta^{js}\mathcal{J}^{rs}(\tau), \qquad \Omega^{oi}(\tau) \approx -\frac{\delta^{ir}\delta^{js}p^j}{p^o + \sqrt{p^2}}\mathcal{J}^{rs}(\tau), \tag{5.30}$$

and the canonical generators of Lorentz transformations become ($\sqrt{p^2} \approx \mathcal{M}$)

$$J^{\mu\nu}(\tau) = q^\mu(\tau)p^\nu(\tau) - q^\nu(\tau)p^\mu(\tau) +$$

$$+ \left[\eta^\mu_i\eta^\nu_j\delta^{ir}\delta^{js} - (\eta^\mu_o\eta^\nu_j - \eta^\nu_o\eta^\mu_j)\frac{\delta^{ir}\delta^{js}p^j}{p^o + \sqrt{p^2}}\right]\mathcal{J}^{rs}(\tau). \tag{5.31}$$

This form of the canonical generator of Lorentz transformations tell us that on a function dependent on the fluid variables alone $F(\vec{\Sigma}(\tau,\vec{\sigma}_o), \vec{K}(\tau,\vec{\sigma}_o))$ a Lorentz transformation acts as a rotation inside the hyperplane. This rotations is the Wigner rotation associated to the infinitesimal Lorentz transformation

$$\delta F = \delta\omega_{\mu\nu}\{F, J^{\mu\nu}\}^{**} = \delta\varphi^{rs}\{F, \mathcal{J}^{rs}\}^{**}, \tag{5.32}$$

where

$$\delta\varphi^{rs} = \left[\eta^\mu_i\eta^\nu_j\delta^{ir}\delta^{js} - (\eta^\mu_o\eta^\nu_j - \eta^\nu_o\eta^\mu_j)\frac{\delta^{ir}\delta^{js}p^j}{p^o + \sqrt{p^2}}\right]\delta\omega_{\mu\nu}. \tag{5.33}$$

The canonical generators of Poincaré group $p^\mu(\tau)$ and $J^{\mu\nu}(\tau)$ are called *external* and they realize the *true* Poincaré symmetry in the reduced phase space. Because of property (5.32), the *Lorentz covariance* of the theory is replaced with the *Wigner covariance* of the 3-dimensional instant form variables on the hyper-planes. In this sense the only canonical variable with a non covariant transformation property is the pseudo four-vector q^μ. As said in Ref. [14], the centroid q^μ is interpreted as a *non-covariant, but canonical external 4-center of mass*. In particular its spatial components are proportional to a 3-center-of-mass-like position that is the classical analogue of the Newton-Wigner position operator, whose reduced covariance corresponds to the little group of time-like Poincare' orbits.

For these particular properties the hyper-planes defined by the condition (5.17) have been called *Wigner hyper-planes* in [8].

We can also construct another realization of the Poincaré Lie algebra. This realization, called *internal*, is constructed using only the fluid variables $\vec{\Sigma}(\tau,\vec{\sigma}_o), \vec{K}(\tau,\vec{\sigma}_o)$ living inside the Wigner hyperplane. In fact, from Eqs. (5.21), (5.22) and (5.23) the 10 functions: $\vec{\mathcal{P}}, \mathcal{M}, \vec{\mathcal{J}}, \vec{\mathcal{K}}$, where

$$\mathcal{J}^r = \frac{1}{2}\epsilon^{ruv}\mathcal{J}^{uv}, \qquad \mathcal{K}^r = \mathcal{J}^{\tau r}, \tag{5.34}$$

are the generators of the following canonical realization of the Poincaré Lie algebra

$$
\begin{aligned}
&\{\mathcal{J}^r, \mathcal{J}^s\} = \epsilon^{rsu}\mathcal{J}^u,\\
&\{\mathcal{K}^r, \mathcal{K}^s\} = -\epsilon^{rsu}\mathcal{J}^u,\\
&\{\mathcal{K}^r, \mathcal{J}^s\} = \epsilon^{rsu}\mathcal{K}^u,\\
&\{\mathcal{P}^r, \mathcal{J}^s\} = \epsilon^{rsu}\mathcal{P}^u,\\
&\{\mathcal{P}^r, \mathcal{K}^s\} = -\mathcal{M}\,\delta^{rs},\\
&\{\mathcal{M}, \mathcal{K}^s\} = -\mathcal{P}^s,\\
&\{\mathcal{M}, \mathcal{P}^s\} = \{\mathcal{P}^r, \mathcal{P}^s\} = 0.
\end{aligned}
\tag{5.35}
$$

This internal realization is unfaithful due to the constraints $\vec{\mathcal{P}} \approx 0$. These constraints says that three degrees of freedom, playing the role of an *internal* 3-center of mass of the fluid inside the Wigner hyperplane, are gauge variables. As shown in Ref. [14], $\vec{\mathcal{K}} \approx 0$ are the natural gauge fixings to eliminate the internal 3-center of mass and to imply $\vec{\lambda}(\tau) = 0$ in the Dirac Hamiltonian of Eq.(5.20). Only the invariant mass $\mathcal{M}$ and the rotation canonical generator $\vec{\mathcal{J}}$ are non vanishing and they appear in the external Poincaré generators (5.31).

With the gauge fixings $\vec{\mathcal{K}} \approx 0$, the Dirac Hamiltonian is reduced to $H_D = \lambda(\tau)\,\mathcal{H}$.

To complete the definition of the *rest-frame instant form* we identify the temporal parameter τ with the common Lorentz scalar rest frame time of the centroids q^μ and x^μ, $T(\tau) = n_\mu x^\mu(\tau) = n_\mu q^\mu(\tau)$. This is realized by introducing the gauge fixing

$$
T(\tau) \approx \tau, \tag{5.36}
$$

implying $\lambda(\tau) = -1$. By using the canonical transformation

$$
\begin{aligned}
&T(\tau) = n_\mu q^\mu(\tau), \qquad \mathcal{E} = \sqrt{p^2},\\
&\vec{z}(\tau) \equiv \sqrt{p^2}\cdot\vec{Q}(\tau) = \sqrt{p^2}\left[\vec{q}(\tau) - \frac{\vec{p}}{p^o}q^o\right], \qquad \vec{k} = \frac{\vec{p}}{p^o},
\end{aligned}
\tag{5.37}
$$

with inverse

$$
\begin{aligned}
&q^o(\tau) = \sqrt{1+\vec{k}^2}\left(T(\tau) + \frac{\vec{k}\cdot\vec{z}(\tau)}{\mathcal{E}}\right), \quad \vec{q}(\tau) = \frac{\vec{z}}{\mathcal{E}} + \left(T(\tau) + \frac{\vec{k}\cdot\vec{z}(\tau)}{\mathcal{E}}\right)\vec{k},\\
&p^o = \mathcal{E}\sqrt{1+\vec{k}^2}, \qquad \vec{p} = \mathcal{E}\vec{k},
\end{aligned}
\tag{5.38}
$$

we arrive at the following Darboux canonical basis [10]

$$\boxed{\begin{array}{c} q^\mu(\tau) \\ p^\mu(\tau) \end{array}} \longrightarrow \boxed{\begin{array}{c|c} T(\tau) & \vec{z}(\tau) \\ \mathcal{E}(\tau) & \vec{k}(\tau) \end{array}}. \tag{5.39}$$

In the rest frame instant form we have $T \approx \tau$, $\mathcal{E} \approx \mathcal{M}$, $H_D = 0$ and the external canonical non-covariant 4-center of mass q^μ is interpreted as a *decoupled point particle clock*. However, since the gauge fixing $T - \tau \approx 0$ is explicitly τ-dependent we get that the effective Hamiltonian to reproduce the Hamilton equations for $\vec{\Sigma}(\tau, \vec{\sigma}_o), \vec{K}(\tau, \vec{\sigma}_o)$ when $\lambda(\tau) = -1$ is

$$H = \mathcal{M} - \vec{\lambda}(\tau) \cdot \vec{\mathcal{P}}. \tag{5.40}$$

In the new canonical variables (5.37) we have $[\Omega^i = (1/2)\epsilon^{ijk}\Omega^{jk}$ with Ω^{jk} given by (5.30)]

$$\vec{J}(\tau) = \vec{z}(\tau) \times \vec{k}(\tau) + \vec{\Omega}(\tau) = \vec{Q}(\tau) \times \vec{p}(\tau) + \vec{\Omega}(\tau). \tag{5.41}$$

The reduced phase space now is

$$\boxed{\begin{array}{c|c} \vec{z}(\tau) & \vec{\Sigma}(\tau, \vec{\sigma}_o) \\ \vec{k}(\tau) & \vec{K}(\tau, \vec{\sigma}_o) \end{array}}. \tag{5.42}$$

The final result is a new instant form of dynamic [25], the *Wigner-covariant rest frame instant form*. In this form the system is described by a reduced phase space (5.42) formed by two sectors:

i) the sector of the *external decoupled point particle clock* described by the canonical non-covariant variables $\vec{z}(\tau), \vec{k}(\tau)$;

ii) the sector of the *internal Wigner-covariant variables* $\vec{\Sigma}(\tau, \vec{\sigma}_o), \vec{K}(\tau, \vec{\sigma}_o)$ (they are Wigner spin-1 3-vectors), living inside the Wigner hyper-planes. Since they are restricted by the constraints $\vec{\mathcal{P}} \approx 0$, $\vec{\mathcal{K}} \approx 0$, only variables relative to an internal inessential 3-center of mass (see next Section) are physical.

---

[10] $\vec{Q} = \vec{z}/\mathcal{E}$ is the classical analogue of the non-covariant Newton-Wigner position operator. However we cannot replace $\vec{z}$ and $\vec{k}$ with $\vec{Q}$, $\vec{p}$, because they do not have vanishing Poisson bracket with the internal variables $\vec{\Sigma}(\tau, \vec{\sigma}_o)$ and $\vec{K}(\tau, \vec{\sigma}_o)$.

The external variables define the 4-unit vector $n^\mu = (\sqrt{1+\vec{k}^2}, \vec{k})$, which identifies the direction of the total four-momentum with respect to an external inertial observer.

In the rest-frame instant form of dynamic the 10 canonical generator of the external Poincaré group are

$$
\begin{aligned}
p^o &= \mathcal{M}\,\sqrt{1+\vec{k}^2} = \sqrt{\mathcal{M}^2+\vec{p}^2},\\
p^i &= \mathcal{M} k^i, \qquad \vec{J} = \vec{z}\times\vec{k} + \vec{\Omega} = \vec{Q}\times\vec{p} + \vec{\Omega},\\
\vec{K} &= -\vec{z}\,\sqrt{1+\vec{k}^2} - \frac{\vec{k}\times\vec{\Omega}}{1+\sqrt{1+\vec{k}^2}} =\\
&= -\vec{Q}\,\sqrt{\mathcal{M}^2+\vec{p}^2} - \frac{\vec{p}\times\vec{\Omega}}{\mathcal{M}+\sqrt{\mathcal{M}^2+\vec{p}^2}},
\end{aligned}
\tag{5.43}
$$

where

$$
\Omega^i \equiv \epsilon^{ijk}\,\delta^{jr}\,\delta^{ks}\left(\frac{1}{2}\epsilon^{rsu}\mathcal{J}^u\right),
\tag{5.44}
$$

is the (interaction-free) spin with respect to the external center of mass.

The properties of the instant form are now shown explicitly: i) $\vec{p}$, $\vec{J}$ do not depend on the interactions; ii) only the 4 generators p^o, $\vec{K}$ depend on the dynamics through $\mathcal{M}$. Even if in a generic instant form the dynamics is determined by four independent potentials (the dynamical $SU(2)$ of Ref. [26]), in the rest frame instant form there is a unique function, the invariant mass $\mathcal{M}$, carrying the whole dynamical information.

Let us observe that the *external* boost generator $K^r = J^{or}$ of Eqs.(5.31) can be rewritten either in the form

$$
\vec{K}(\tau) = q^o(\tau)\vec{p} - \vec{q}(\tau)p^o - \frac{\vec{p}\times\vec{\Omega}(\tau)}{p^o+\sqrt{p^2}},
\tag{5.45}
$$

or in the form

$$
\vec{K}(\tau) = q^o(\tau)\vec{p} + \vec{K}'(\tau).
\tag{5.46}
$$

If $\vec{Q}(\tau)$ is defined by (5.37), we have

$$
\vec{Q}(\tau) = -\frac{\vec{K}(\tau)}{p^o} - \frac{\vec{p}\times\vec{\Omega}(\tau)}{p^o(p^o+\sqrt{p^2})},
\tag{5.47}
$$

and

$$\vec{q}(\tau) = -\frac{\vec{K}'(\tau)}{p^o} - \frac{\vec{p}\times\vec{\Omega}(\tau)}{p^o(p^o+\sqrt{p^2})}. \tag{5.48}$$

In this way we recover the usual definitions of the canonical, non-covariant relativistic 3-center of mass [27]. The first is given using the complete external boost generator, the second using the boost generator on the hyperplane $q^o(\tau)=0$. Being $n_\mu x^\mu(\tau) = n_\mu q^\mu(\tau)$, the 4-center of mass $q^\mu(\tau)$ defines a point on the Wigner hyperplane different from the centroid $x^\mu(\tau) = z^\mu(\tau,\vec{0})$, with coordinates $\vec{\sigma}(q) \neq 0$ such that $q^\mu(\tau) = z^\mu(\tau,\vec{\sigma}(q))$. Using only the canonical generators of the external realization of the Poincaré group it is also possible to define the external non-covariant, non-canonical *Møller 3-center of energy* [27]:

$$\vec{R}(\tau) = -\frac{\vec{K}(\tau)}{p^o}, \tag{5.49}$$

and the external covariant, non-canonical relativistic *Fokker-Pryce 3-center of inertia* [18]:

$$\vec{Y}(\tau) = -\frac{\vec{K}(\tau)}{p^o} - \frac{\vec{p}\times\vec{\Omega}(\tau)}{p^o\sqrt{p^2}}. \tag{5.50}$$

In Ref. [14] it is shown how to identify the position on the Wigner hyperplane of the associated external (pseudo-vector) 4-center of energy R^μ and of the external 4-center of inertia Y^μ, which is a 4-vector by construction. If we put $q^\mu = (q^o;\vec{q})$, then we have $R^\mu = (q^o;\vec{R}+q^o\,\vec{p})$ and $Y^\mu = (q^o;\vec{Y}+q^o\,\vec{p})$. In Ref. [14] it is also shown that all the possible pseudo-vectors q^μ and R^μ fill a world-tube around the 4-vector Y^μ, whose radius is the *Møller radius* [28] $\rho = \frac{|\vec{\Omega}|}{\mathcal{M}}$ of the fluid configuration. This radius is defined by the Poincare' Casimirs of the external Poincare' group ($p^2 = \mathcal{M}^2$, $W^2 = -\mathcal{M}^2\,\vec{\Omega}^2$) and is a classical unit of length determined by the Cauchy data of the configuration of the system. See Ref. [3] for the properties of this radius and for the proposal of using it as a natural physical ultraviolet cutoff at the quantum level for all rotating configurations of the isolated system.

Let us conclude this Section with the Hamilton equations associated with the Hamiltonian (5.40) in a gauge where $\vec{\lambda}(\tau)=0$

$$\begin{aligned}
\frac{\partial\vec{\Sigma}(\tau,\vec{\sigma}_o)}{\partial\tau} &\circeq \{\vec{\Sigma}(\tau,\vec{\sigma}_o),\mathcal{M}\},\\
\frac{\partial\vec{K}(\tau,\vec{\sigma}_o)}{\partial\tau} &\circeq \{\vec{K}(\tau,\vec{\sigma}_o),\mathcal{M}\}.
\end{aligned} \tag{5.51}$$

For the dust Eq.(B6) implies

$$\frac{\partial \vec{\Sigma}(\tau, \vec{\sigma}_o)}{\partial \tau} \circeq \frac{\vec{K}(\tau, \vec{\sigma}_o)}{\sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 + \vec{K}^2(\tau, \vec{\sigma}_o)}}, \qquad \frac{\partial \vec{K}(\tau, \vec{\sigma}_o)}{\partial \tau} \circeq 0,$$

$$\vec{\Sigma}(\tau, \vec{\sigma}_o) \circeq \vec{\sigma}_o + \frac{\vec{K}(\tau, \vec{\sigma}_o)}{\sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 + \vec{K}^2(\tau, \vec{\sigma}_o)}}\, \tau. \tag{5.52}$$

Let us remark that in this description these hyperbolic Hamilton equations replace the hydrodynamical Euler equations (2.26) implied by the conservation (2.23) of the stress-energy tensor.

VI. INTERNAL CENTERS OF MASS AND RELATIVE VARIABLES

In the rest frame instant form defined in the previous Section the 3-dimensional variables on the Wigner hyper-planes are not all *physical* due to the first class Dirac constraint $\vec{\mathcal{P}} \approx 0$. To select the physical degree of freedom is a problem equivalent to determine the internal 3-center of mass and relative variables on the Wigner hyperplane. As already said, the internal 3-center of mass variable on the Wigner hyperplane is a gauge variable, because the role of true center of mass is played by the *external*, canonical non-covariant, 3-center of mass $\vec{z}$. On the contrary the relative variables will be the physical variables and they will describe the reduced phase space after the final gauge fixings, whose natural form is $\vec{\mathcal{K}} \approx 0$.

To justify the gauge fixing $\vec{\mathcal{K}} \approx 0$ we must perform two steps. First we select a naive internal center of mass position $\vec{\mathcal{X}}$ canonical with respect to the total momentum $\vec{\mathcal{P}}$ and the associated canonical relative variables. The naive center of mass position $\vec{\mathcal{X}}$ allows to define the gauge fixing $\vec{\mathcal{X}} \approx 0$, but this gauge fixing has the unpleasant property that the arbitrary Dirac multiplier $\vec{\lambda}(\tau)$ is fixed to a non null value ($\vec{\lambda}(\tau) \neq 0$). Then we use the internal relative variables obtained as auxiliary variables in the first step for defining the relative variable respect to a 3-center of mass $\vec{\mathcal{Q}}$ such that the gauge fixings $\vec{\mathcal{Q}} \approx 0$ (identification of the internal 3-center of mass with the centroid $x^\mu(\tau)$ origin of the 3-coordinates) imply $\vec{\lambda}(\tau) \approx 0$. This is done using the method of the *Gartenhaus-Schwartz* transformation of Ref. [29] (see also Appendix D).

Let the total internal 3-momentum and an internal naive 3-center-of-mass position on the Wigner hyperplane $\vec{\mathcal{X}}$ be defined as

$$\mathcal{P}^r(\tau) = \int d^3\sigma_o\, K^r(\tau, \vec{\sigma}_o),$$

$$\mathcal{X}^s(\tau) = \frac{1}{\mathcal{N}} \int d^3\sigma_o\, n_o(\vec{\sigma}_o) \Sigma^r(\tau, \vec{\sigma}_o), \tag{6.1}$$

with

$$\mathcal{N} = \int d^3\sigma_o\, n_o(\vec{\sigma}_o), \tag{6.2}$$

and

$$\{\mathcal{X}^r(\tau), \mathcal{P}^s(\tau)\} = \delta^{rs}. \tag{6.3}$$

Let the *internal relative canonical variables* $\Re^r(\tau, \vec{\sigma}_o), \wp^s(\tau, \vec{\sigma}_o)$ be defined in such a way that we get

$$\Sigma^r(\tau, \vec{\sigma}_o) = \mathcal{X}^r(\tau) + \int d^3\sigma'_o\, \Gamma_\Sigma(\vec{\sigma}_o, \vec{\sigma}'_o) \Re^r(\tau, \vec{\sigma}'_o),$$

$$K^s(\tau, \vec{\sigma}_o) = \frac{n_o(\vec{\sigma}_o)}{\mathcal{N}} \mathcal{P}^r(\tau) + \int d^3\sigma'_o\, \wp^r(\tau, \vec{\sigma}'_o) \Gamma_K(\vec{\sigma}'_o, \vec{\sigma}_o). \tag{6.4}$$

The kernels Γ_K, Γ_Σ will be specified by some conditions that we will analyze shortly. From the definitions (6.1) we obtain that

$$\int d^3\sigma_o\, n_o(\vec{\sigma}_o) \Gamma_\Sigma(\vec{\sigma}_o, \vec{\sigma}'_o) = 0,$$

$$\int d^3\sigma_o\, \Gamma_K(\vec{\sigma}'_o, \vec{\sigma}_o) = 0. \tag{6.5}$$

We impose the following canonical property

$$\{\Re^r(\tau, \vec{\sigma}_o), \wp^s(\tau, \vec{\sigma}'_o)\} = \delta^{rs} \delta(\vec{\sigma}_o - \vec{\sigma}'_o). \tag{6.6}$$

By using Eq.(6.3), we can verify that from Eq.(6.4) we obtain the canonical property

$$\{\Sigma^r(\tau, \vec{\sigma}_o), K^s(\tau, \vec{\sigma}'_o)\} = \delta^{rs}\, \delta(\vec{\sigma}_o - \vec{\sigma}'_o), \tag{6.7}$$

if

$$\int d^3\sigma_o\, \Gamma_\Sigma(\vec{\sigma}_{o1}, \vec{\sigma}) \Gamma_K(\vec{\sigma}, \vec{\sigma}_{o2}) = -\frac{n_o(\vec{\sigma}_{o2})}{\mathcal{N}} + \delta(\vec{\sigma}_{o1} - \vec{\sigma}_{o2}). \tag{6.8}$$

When Eqs.(6.8) hold, Eq.(6.4) is consistent with the following definitions

$$\Re^r(\tau, \vec{\sigma}_o) = \int d^3\sigma'_o\, \Gamma_K(\vec{\sigma}_o, \vec{\sigma}'_o) \Sigma^r(\tau, \vec{\sigma}'_o),$$

$$\wp^r(\tau, \vec{\sigma}_o) = \int d^3\sigma'_o\, K^r(\tau, \vec{\sigma}'_o) \Gamma_\Sigma(\vec{\sigma}'_o, \vec{\sigma}_o). \tag{6.9}$$

In fact, if we substitute these expression in Eq.(6.4), using Eq.(6.8) we have an identity. Again from Eq.(6.9) and using Eq.(6.7), we can find Eq.(6.6) if the following condition is verified

$$\int d^3\sigma_o\, \Gamma_K(\vec{\sigma}_{o1}, \vec{\sigma}_o) \Gamma_\Sigma(\vec{\sigma}_o, \vec{\sigma}_{o2}) = \delta(\vec{\sigma}_{o1} - \vec{\sigma}_{o2}). \tag{6.10}$$

Eqs. (6.5),(6.8) and (6.10) are a set of conditions that have to be satisfied by the kernels Γ. These conditions are not independent: it can be proved that Eqs. (6.8) e (6.10) imply Eq.(6.5).

We can also verify that, using Eq.(6.5), we have

$$\vec{\mathcal{J}}(\tau) = \vec{\mathcal{X}}(\tau) \times \vec{\mathcal{P}}(\tau) + \int d^3\sigma_o \, \vec{\Re}(\tau, \vec{\sigma}_o) \times \vec{\wp}(\tau, \vec{\sigma}_o),$$

$$\vec{\mathcal{K}}(\tau) = -\mathcal{M}(\tau) \, \vec{\mathcal{X}}(\tau) - \int d^3\sigma_o \, d^3\sigma_o^{'} \, \Gamma_\Sigma(\vec{\sigma}_o, \vec{\sigma}_o^{'}) \, \vec{\Re}(\tau, \vec{\sigma}_o^{'}) \, \Delta(\tau, \vec{\sigma}_o). \tag{6.11}$$

Let $\Phi_{\underline{n}}(\vec{\sigma}_o)$ be a base of orthonormal functions on R^3 with $\underline{n} = (n_1, n_2, n_3)$ a set of multindices. Then we can consider the coefficients

$$\vec{r}_{\underline{n}}(\tau) = \int d^3\sigma_o \, \Phi_{\underline{n}}(\vec{\sigma}_o) \vec{\Re}(\tau, \vec{\sigma}_o),$$

$$\vec{p}_{\underline{n}}(\tau) = \int d^3\sigma_o \, \Phi_{\underline{n}}(\vec{\sigma}_o) \vec{\wp}(\tau, \vec{\sigma}_o), \tag{6.12}$$

such that

$$\vec{\Re}(\tau, \vec{\sigma}_o) = \sum_{\underline{n}} \vec{r}_{\underline{n}}(\tau) \, \Phi_{\underline{n}}(\vec{\sigma}_o),$$

$$\vec{\wp}(\tau, \vec{\sigma}_o) = \sum_{\underline{n}} \vec{p}_{\underline{n}}(\tau) \, \Phi_{\underline{n}}(\vec{\sigma}_o). \tag{6.13}$$

Moreover from Eq.(6.11) we get

$$\int d^3\sigma_o \, \vec{\Re}(\tau, \vec{\sigma}_o) \times \vec{\wp}(\tau, \vec{\sigma}_o) = \sum_{\underline{n}} \vec{r}_{\underline{n}}(\tau) \times \vec{p}_{\underline{n}}(\tau), \tag{6.14}$$

and

$$\{ r^r_{\underline{n}}(\tau), p^s_{\underline{m}}(\tau) \} = \delta^{rs} \delta_{\underline{nm}}. \tag{6.15}$$

In conclusion the coefficients $\vec{r}_{\underline{n}}(\tau)$, $\vec{p}_{\underline{n}}(\tau)$ are a set of infinite canonical variables that we can use as *internal relative variables*. These variables are useful for defining the canonical transformation that will realize the separation between rotational and shape degree of freedom in the next Section. For the time being we can use them to rewrite the definitions (6.4),(6.9) in the form

$$\vec{\Sigma}(\tau,\vec{\sigma}_o) = \vec{\mathcal{X}}(\tau) + \sum_{\underline{n}} \Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o)\, \vec{r}_{\underline{n}}(\tau),$$

$$\vec{K}(\tau,\vec{\sigma}_o) = \frac{n_o(\vec{\sigma}_o)}{\mathcal{N}} \vec{\mathcal{P}}(\tau) + \sum_{\underline{n}} \Gamma^{K}_{\underline{n}}(\vec{\sigma}_o)\, \vec{p}_{\underline{n}}(\tau),$$

$$\vec{r}_{\underline{n}}(\tau) = \int d^3\sigma_o\, \Gamma^{K}_{\underline{n}}(\vec{\sigma}_o)\, \vec{\Sigma}(\tau,\vec{\sigma}_o),$$

$$\vec{p}_{\underline{n}}(\tau) = \int d^3\sigma_o\, \Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o)\, \vec{K}(\tau,\vec{\sigma}_o), \qquad (6.16)$$

where

$$\Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o) = \int d^3\sigma'_o\, \Gamma_{\Sigma}(\vec{\sigma}_o,\vec{\sigma}'_o)\Phi_{\underline{n}}(\vec{\sigma}'_o),$$

$$\Gamma^{K}_{\underline{n}}(\vec{\sigma}_o) = \int d^3\sigma'_o\, \Phi_{\underline{n}}(\vec{\sigma}'_o)\Gamma_K(\vec{\sigma}'_o,\vec{\sigma}_o). \qquad (6.17)$$

The conditions (6.5) and (6.8), (6.10) on the kernel functions Γ are rewritten in the form

$$\int d^3\sigma_o\, n_o(\vec{\sigma}_o)\, \Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o) = 0, \qquad \int d^3\sigma_o\, \Gamma^{K}_{\underline{n}}(\vec{\sigma}_o) = 0,$$

$$\sum_{\underline{n}} \Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_{1o})\Gamma^{K}_{\underline{n}}(\vec{\sigma}_{2o}) = -\frac{n_o(\vec{\sigma}_{2o})}{\mathcal{N}} + \delta(\vec{\sigma}_{1o} - \vec{\sigma}_{2o}),$$

$$\int d^3\sigma_o\, \Gamma^{K}_{\underline{n}}(\vec{\sigma}_o)\, \Gamma^{\Sigma}_{\underline{m}}(\vec{\sigma}_o) = \delta_{\underline{nm}}. \qquad (6.18)$$

Some possible solutions for the kernels Γ are derived in Appendix E.

As said at the beginning of this Section, the internal 3-center of mass-like position $\vec{\mathcal{X}}$ is such that the gauge fixing $\vec{\mathcal{X}} \approx 0$ does not imply $\vec{\lambda}(\tau) \approx 0$ as can be checked by using the Dirac Hamiltonian $H_D = \mathcal{M} - \vec{\lambda}(\tau)\cdot\vec{\mathcal{P}}$ in the gauge $T \approx \tau$. We want to replace it with another internal 3-center of mass $\vec{\mathcal{Q}}$ such that the conditions $\vec{\mathcal{Q}} \approx 0$ imply $\vec{\lambda}(\tau) \approx 0$.

To this end we construct the *internal* 3-centers of mass, energy and inertia in analogy to the *external* ones of the previous Section, using the *internal* realization of the Poincaré algebra instead of the external one

$$\vec{\mathcal{R}}(\tau) = -\frac{\vec{\mathcal{K}}(\tau)}{\mathcal{M}},$$

$$\vec{\mathcal{Q}}(\tau) = -\frac{\vec{\mathcal{K}}(\tau)}{\mathcal{M}} - \frac{\vec{\mathcal{P}} \times \vec{\mathcal{S}}(\tau)}{\mathcal{M}(\mathcal{M} + \sqrt{\mathcal{M}^2 - \vec{\mathcal{P}}^2})} \approx \vec{\mathcal{R}}(\tau),$$

$$\vec{\mathcal{Y}}(\tau) = -\frac{\vec{\mathcal{K}}(\tau)}{\mathcal{M}} - \frac{\vec{\mathcal{P}} \times \vec{\mathcal{S}}(\tau)}{\mathcal{M}\sqrt{\mathcal{M}^2 - \vec{\mathcal{P}}^2}} \approx \vec{\mathcal{R}}(\tau), \tag{6.19}$$

where

$$\vec{\mathcal{J}}(\tau) = \vec{\mathcal{Q}} \times \vec{\mathcal{P}} + \vec{\mathcal{S}}(\tau),$$

$$\vec{\mathcal{K}}(\tau) = -\mathcal{M}(\tau)\, \vec{\mathcal{X}}(\tau) - \sum_{\underline{n}} \vec{r}_{\underline{n}}(\tau) \int d^3\sigma_o\, \Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o)\, \Delta(\tau, \vec{\sigma}_o) =$$

$$= -\mathcal{M}(\tau)\, \vec{\mathcal{R}}(\tau) \approx -\mathcal{M}(\tau)\, \vec{\mathcal{Q}}(\tau),$$

$$\Downarrow$$

$$\vec{\mathcal{X}}(\tau) = \vec{\mathcal{R}}(\tau) - \sum_{\underline{n}} \frac{\vec{r}_{\underline{n}}(\tau)}{\mathcal{M}(\tau)} \int d^3\sigma_o\, \Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o)\, \Delta(\tau, \vec{\sigma}_o). \tag{6.20}$$

These three centers are weakly equal due to the constraint $\vec{\mathcal{P}} \approx 0$ and they are all canonically conjugate to $\vec{\mathcal{P}}$

$$\{\mathcal{R}^r, \mathcal{P}^s\}^{**} = \{\mathcal{Q}^r, \mathcal{P}^s\}^{**} = \{\mathcal{Y}^r, \mathcal{P}^s\}^{**} = \delta^{rs}, \tag{6.21}$$

and such that

$$\{\mathcal{R}^r, \mathcal{M}\}^{**} = \{\mathcal{Q}^r, \mathcal{M}\}^{**} = \{\mathcal{Y}^r, \mathcal{M}\}^{**} = \frac{\mathcal{P}^r}{\mathcal{M}} \approx 0. \tag{6.22}$$

But $\vec{\mathcal{Q}}$ is the only one such that

$$\{\mathcal{Q}^r, \mathcal{Q}^s\}^{**} = 0, \tag{6.23}$$

namely it is the real internal canonical 3-center of mass.

If we adopt the gauge fixings

$$\vec{\mathcal{Q}}(\tau) \approx \vec{\mathcal{R}}(\tau) \approx \vec{\mathcal{Y}}(\tau) \approx 0, \tag{6.24}$$

then we get

$$
\begin{aligned}
\dot{\vec{\mathcal{Q}}}(\tau) &\stackrel{\circ}{=} \{\vec{\mathcal{Q}}(\tau), \mathcal{M} - \vec{\lambda}(\tau)\cdot\vec{\mathcal{P}}\} = -\vec{\lambda}(\tau) \approx 0,\\
\vec{\mathcal{X}}(\tau) &\approx -\sum_{\underline{n}} \frac{\vec{r}_{\underline{n}}(\tau)}{\mathcal{M}(\tau)} \int d^3\sigma_o\, \Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o)\, \Delta(\tau,\vec{\sigma}_o)\\
\vec{\Sigma}(\tau,\vec{\sigma}_o) &\approx \sum_{\underline{n}} \vec{r}_{\underline{n}}(\tau) \left[\Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o) - \frac{\int d^3\sigma'_o\, \Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}'_o)\, \Delta(\tau,\vec{\sigma}'_o)}{\mathcal{M}(\tau)}\right] =\\
&\stackrel{def}{=} \sum_{\underline{n}} \vec{r}_{\underline{n}}(\tau) \left[\Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o) - h_{\underline{n}}\right].
\end{aligned}
\tag{6.25}
$$

Eq.(6.19) shows that $\vec{\mathcal{Q}}(\tau) \approx 0$ is equivalent to the condition $\vec{\mathcal{K}} \approx 0$. After this gauge fixing, in the internal unfaithful realization of the Poincaré group there are only four non null functions: $\mathcal{M}$ and $\vec{\mathcal{J}}$.

In Appendix D it is show that with the relative variables $\Re^r(\tau,\vec{\sigma}_o), \wp^s(\tau,\vec{\sigma}_o)$ we can realize the *Garthenaus-Schwartz* canonical transformation

$$
\boxed{\begin{array}{c} \vec{\Sigma}(\tau,\vec{\sigma}_o) \\ \vec{K}(\tau,\vec{\sigma}_o) \end{array}} \longrightarrow \boxed{\begin{array}{c|c} \vec{\mathcal{Q}}(\tau) & \Re'^r(\tau,\vec{\sigma}_o) \\ \vec{\mathcal{P}}(\tau) & \wp'^s(\tau,\vec{\sigma}_o) \end{array}}.
\tag{6.26}
$$

In Appendix D it is also shown that, after having gone to Dirac brackets with respect to the second class constraints $\vec{\mathcal{P}} \approx 0$, $\vec{\mathcal{Q}} \approx 0$, we get

$$
\vec{\mathcal{P}} \equiv \vec{\mathcal{Q}} \equiv 0 \quad \Rightarrow \quad \Re'^r(\tau,\vec{\sigma}_o) \equiv \Re^r(\tau,\vec{\sigma}_o), \quad \wp'^s(\tau,\vec{\sigma}_o) \equiv \wp^s(\tau,\vec{\sigma}_o).
\tag{6.27}
$$

Then the final reduced phase space is

$$
\boxed{\begin{array}{c|c} \vec{z}(\tau) & \Re^r(\tau,\vec{\sigma}_o) \\ \vec{k}(\tau) & \wp^r(\tau,\vec{\sigma}_o) \end{array}},
\tag{6.28}
$$

where

$$
\vec{\mathcal{J}}(\tau) \approx \vec{\mathcal{S}}(\tau) = \int d^3\sigma_o\, \vec{\Re}(\tau,\vec{\sigma}_o) \times \vec{\wp}(\tau,\vec{\sigma}_o) = \sum_{\underline{n}} \vec{r}_{\underline{n}}(\tau) \times \vec{p}_{\underline{n}}(\tau).
\tag{6.29}
$$

On the Wigner hyper-planes the kinetic term $\vec{K}^2(\tau,\vec{\sigma}_o)$ appearing in the solution $X(\tau,\vec{\sigma}_o)$ of Eq.(4.28) is

$$\vec{K}^2(\tau, \vec{\sigma}_o) \approx \sum_{\underline{n}_1 \underline{n}_2} \Gamma^K_{\underline{n}_1}(\vec{\sigma}_o)\, \Gamma^K_{\underline{n}_2}(\vec{\sigma}_o)\, \vec{p}_{\underline{n}_1}(\tau) \cdot \vec{p}_{\underline{n}_2}(\tau), \tag{6.30}$$

while the dependence of $X(\tau, \vec{\sigma}_o)$ on the generalized Eulerian coordinates is concentrated in

$$\det \left(\frac{\partial \Sigma}{\partial \sigma_o} \right) = \det \left(\sum_{\underline{n}} \frac{\partial\, \Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o)}{\partial \sigma^s_o}\, r^r_{\underline{n}}(\tau) \right). \tag{6.31}$$

VII. ROTATIONAL KINEMATICS

The internal relative canonical variables $\vec{r}_{\underline{n}}(\tau)$, $\vec{p}_{\underline{n}}(\tau)$ are Wigner spin 1 vectors under rotations. Then we can do on them a canonical transformation that generalize the results obtained in the N particles case in Refs. [13,14]. The $\vec{r}_{\underline{n}}$ are interpreted as a set of infinite relative position vectors; we want use them to construct a *dynamical body frame* as done in Refs. [13,14]. For this *we have to select a pair of these vectors.* We assume to choose the vector positions with multindices $\underline{u}_1, \underline{u}_2$ as preferred vectors

$$\vec{r}_{\underline{u}_i} = \vec{R}_i, \quad \vec{p}_{\underline{u}_i} = \vec{\Pi}_i, \quad i = 1, 2, \tag{7.1}$$

and we use them to define the orthogonal vectors

$$\vec{N} = \frac{\hat{R}_1 + \hat{R}_2}{2}, \qquad \vec{\chi} = \frac{\hat{R}_1 - \hat{R}_2}{2}, \qquad \vec{N} \cdot \vec{\chi} = 0. \tag{7.2}$$

The most convenient choice of these two vectors will be dictated by the spatial form of the initial density $n_o(\vec{\sigma}_o)$.

Then a *dynamical body frame* is defined by the associated unit vectors and their orthogonal complement

$$\hat{b}_r(\tau) = (\hat{\chi}(\tau), \hat{N}(\tau) \times \hat{\chi}(\tau), \hat{N}(\tau)). \tag{7.3}$$

By construction this frame is a orthonormal frame that rotates with the motion of the fluid. In this sense it generalize the concept of *body frame* of a non relativistic rigid body and we can apply the same observations and interpretation done in Refs. [13,14]. Moreover we observe that

$$\{N^r, N^s\} = \{\chi^r, \chi^s\} = \{N^r, \chi^s\} = 0. \tag{7.4}$$

All vectors can be projected on this *dynamical body frame* so to obtain their components on it. In particular, for the relative angular momentum, given by Eq. (6.29), its components on the *dynamical body frame* are

$$\overline{\mathcal{S}}^r(\tau) = \vec{\mathcal{S}}(\tau) \cdot \hat{b}_r(\tau), \tag{7.5}$$

or more explicitly

$$\overline{\mathcal{S}}^1 = \vec{\mathcal{S}} \cdot \hat{\chi}; \qquad \overline{\mathcal{S}}^2 = \vec{\mathcal{S}} \cdot (\hat{N} \times \hat{\chi}); \qquad \overline{\mathcal{S}}^3 = \vec{\mathcal{S}} \cdot \hat{N}. \tag{7.6}$$

Using the results of Refs. [13,14] we can construct the following quantities

$$\vec{\mathcal{W}} = \vec{R}_1 \times \vec{\Pi}_1 - \vec{R}_2 \times \vec{\Pi}_2, \tag{7.7}$$

$$R_i =\mid \vec{R}_i \mid, \qquad \tilde{\Pi}_i = \vec{\Pi}_i \cdot \hat{R}_i, \qquad i = 1, 2, \tag{7.8}$$

and, for $\underline{n} \neq \underline{u}_1, \underline{u}_2$

$$\overline{r}^1_{\underline{n}} = \vec{r}_{\underline{n}} \cdot \hat{\chi}, \quad \overline{r}^2_{\underline{n}} = \vec{r}_{\underline{n}} \cdot \hat{N} \times \hat{\chi}, \quad \overline{r}^3_{\underline{n}} = \vec{r}_{\underline{n}} \cdot \hat{N},$$

$$\overline{p}^1_{\underline{n}} = \vec{p}_{\underline{n}} \cdot \hat{\chi}, \quad \overline{p}^2_{\underline{n}} = \vec{p}_{\underline{n}} \cdot \hat{N} \times \hat{\chi}, \quad \overline{p}^3_{\underline{n}} = \vec{p}_{\underline{n}} \cdot \hat{N}. \tag{7.9}$$

Then the transformation represented in the table

$$\begin{array}{|c|} \hline \vec{r}_{\underline{n}} \\ \\ \vec{p}_{\underline{n}} \\ \hline \end{array} \longrightarrow \begin{array}{|ccc|ccc|c|} \hline \mid \vec{\mathcal{S}} \mid & \mathcal{S}^3 & \overline{\mathcal{S}}^3 & \mid \vec{N} \mid & R_1 & R_2 & \overline{r}^s_{\underline{n} \neq \underline{u}_1, \underline{u}_2} \\ \alpha & \beta & \gamma & \xi & \tilde{\Pi}_1 & \tilde{\Pi}_2 & \overline{p}^s_{\underline{n} \neq \underline{u}_1, \underline{u}_2} \\ \hline \end{array}, \tag{7.10}$$

is a canonical transformation (the canonical pairs are the variables on the same column) if we define

$$\alpha = \tan^{-1} \frac{(\hat{\mathcal{S}} \times \hat{N})^3}{[\hat{\mathcal{S}} \times (\hat{\mathcal{S}} \times \hat{N})]^3},$$

$$\beta = \tan^{-1} \frac{\mathcal{S}^2}{\mathcal{S}^1},$$

$$\gamma = \tan^{-1} \frac{\overline{\mathcal{S}}^2}{\overline{\mathcal{S}}^1},$$

$$\xi = \frac{\vec{\mathcal{W}} \cdot (\hat{N} \times \hat{\chi})}{\sqrt{1 - \vec{N}^2}}. \tag{7.11}$$

The canonical variables in the final basis have been separated in three sectors. The second and the third sector in the previous table are constituted by canonical variables scalar under rotations: these variables describe the *shape* of the fluid and we call them *shape (or vibrational) variables* (see Ref. [30] for the original definition of shape variables). The first sector is that of the *rotational (or orientational) variables*; these variables describe the rotational motion of the dynamical body frame.

It is useful to analyze the variables using a new orthonormal base: the *spin basis* of Refs. [31,13,14]. This basis is defined observing that there is only a unit vector $\hat{\mathcal{R}}$ on the same plane of $\vec{N}$ and $\vec{\mathcal{S}}$, orthogonal to $\vec{\mathcal{S}}$ such that

$$\alpha = \tan^{-1} \frac{(\hat{\mathcal{S}} \times \hat{\mathcal{R}})^3}{[\hat{\mathcal{S}} \times (\hat{\mathcal{S}} \times \hat{\mathcal{R}})]^3}. \tag{7.12}$$

Then the three unit vectors $(\hat{\mathcal{R}}, \hat{\mathcal{S}}, \hat{\mathcal{S}} \times \hat{\mathcal{R}})$ are a orthonormal basis, the *spin basis*. By construction their components are given by the following relations

$$\begin{cases} \hat{\mathcal{S}}^1 = \frac{\sqrt{\vec{\mathcal{S}}^2 - (\mathcal{S}^3)^2}}{|\vec{\mathcal{S}}|} \cos\beta, \\ \hat{\mathcal{S}}^2 = \frac{\sqrt{\vec{\mathcal{S}}^2 - (\mathcal{S}^3)^2}}{|\vec{\mathcal{S}}|} \sin\beta, \\ \hat{\mathcal{S}}^3 = \frac{\mathcal{S}^3}{|\vec{\mathcal{S}}|}, \end{cases} \tag{7.13}$$

$$\begin{cases} \hat{\mathcal{R}}^1 = \sin\beta \sin\alpha - \frac{\mathcal{S}^3}{|\vec{\mathcal{S}}|} \cos\beta \cos\alpha, \\ \hat{\mathcal{R}}^2 = -\cos\beta \sin\alpha - \frac{\mathcal{S}^3}{|\vec{\mathcal{S}}|} \sin\beta \cos\alpha, \\ \hat{\mathcal{R}}^3 = \frac{\sqrt{\vec{\mathcal{S}}^2 - (\mathcal{S}^3)^2}}{|\vec{\mathcal{S}}|} \cos\alpha, \end{cases} \tag{7.14}$$

$$\begin{cases} (\hat{\mathcal{S}} \times \hat{\mathcal{R}})^1 = \sin\beta \sin\alpha + \frac{\mathcal{S}^3}{|\vec{\mathcal{S}}|} \cos\beta \sin\alpha, \\ (\hat{\mathcal{S}} \times \hat{\mathcal{R}})^2 = -\cos\beta \cos\alpha + \frac{\mathcal{S}^3}{|\vec{\mathcal{S}}|} \sin\beta \sin\alpha, \\ (\hat{\mathcal{S}} \times \hat{\mathcal{R}})^3 = \frac{\sqrt{\vec{\mathcal{S}}^2 - (\mathcal{S}^3)^2}}{|\vec{\mathcal{S}}|} \sin\alpha. \end{cases} \tag{7.15}$$

We also define the angle ψ such that

$$\cos\psi = \frac{\overline{\mathcal{S}}^3}{|\,\vec{\mathcal{S}}\,|}; \quad \sin\psi = \frac{\sqrt{\vec{\mathcal{S}}^2 - (\overline{\mathcal{S}}^3)^2}}{|\,\vec{\mathcal{S}}\,|}. \tag{7.16}$$

By definition of $\hat{\mathcal{R}}$ we have

$$\hat{N} = \cos\psi\,\hat{\mathcal{S}} + \sin\psi\,\hat{\mathcal{R}}. \tag{7.17}$$

Moreover the definition of γ implies that we have

$$\begin{aligned} \hat{\mathcal{S}} \cdot \hat{\chi} &= \sin\psi\cos\gamma, \\ \hat{\mathcal{S}} \cdot (\hat{\chi} \times \hat{N}) &= \sin\psi\sin\gamma. \end{aligned} \tag{7.18}$$

We complete the conditions on $\hat{\chi}, \hat{N} \times \hat{\chi}$ using the fact that the *dynamical body frame* and the *spin basis* are connected by a (proper) rotation

$$\begin{aligned} \hat{\chi} &= \sin\psi\cos\gamma\,\hat{\mathcal{S}} - \cos\psi\cos\gamma\,\hat{\mathcal{R}} + \sin\gamma\,\hat{\mathcal{S}} \times \hat{\mathcal{R}}, \\ \hat{\chi} \times \hat{N} &= \sin\psi\sin\gamma\,\hat{\mathcal{S}} - \cos\psi\sin\gamma\,\hat{\mathcal{R}} - \cos\gamma\,\hat{\mathcal{S}} \times \hat{\mathcal{R}}. \end{aligned} \tag{7.19}$$

Substituting in Eqs.(7.17),(7.19) the expression given by Eqs.(7.13),(7.14),(7.15),(7.16) we obtain the elements of the *dynamical body frame* expressed as functions of the *rotational variables* alone. We can also define the *Euler's angles* of the *dynamical body frame* [11] $\varepsilon^1, \varepsilon^2, \varepsilon^3$

$$\cos\varepsilon^2 = \hat{N}^3; \quad \cos\varepsilon^1 = \frac{\hat{N}^1}{\sqrt{1 - (\hat{N}^3)^2}}; \quad \cos\varepsilon^3 = -\frac{\hat{\chi}^3}{\sqrt{1 - (\hat{N}^3)^2}}. \tag{7.20}$$

They are as functions of the *rotational variables* alone.

The Euler's angles (7.20) together with the relative angular momentum components

$$\begin{cases} \overline{\mathcal{S}}^1 = \sqrt{\vec{\mathcal{S}}^2 - (\overline{\mathcal{S}}^3)^2}\cos\gamma, \\ \\ \overline{\mathcal{S}}^2 = \sqrt{\vec{\mathcal{S}}^2 - (\overline{\mathcal{S}}^3)^2}\sin\gamma, \\ \\ \overline{\mathcal{S}}^3. \end{cases} \tag{7.21}$$

define a non canonical transformation for the rotational sector. The corresponding canonical transformation is obtained using the canonical momenta

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[11] As in Refs. [13,14] we adopt the y-convention of Ref. [32]

$$p_1 = -\sin\varepsilon^2 \cos\varepsilon^3\, \overline{\mathcal{S}}^1 + \sin\varepsilon^2\ \sin\varepsilon^3\, \overline{\mathcal{S}}^2 + \cos\varepsilon^2\, \overline{\mathcal{S}}^3,$$

$$p_2 = \sin\varepsilon^3\, \overline{\mathcal{S}}^1 + \cos\varepsilon^3\, \overline{\mathcal{S}}^2, \qquad p_3 = \overline{\mathcal{S}}^3,$$

$$\{\varepsilon^r, \varepsilon^s\} = \{p_r, p_s\} = 0, \qquad \{\varepsilon^r, p_s\} = \delta^r_s. \tag{7.22}$$

The inverses of the previous equations are

$$\overline{\mathcal{S}}^1 = \sin\varepsilon^3\, p_2 - \frac{\cos\varepsilon^3}{\sin\varepsilon^2}\, p_1 + \cos\varepsilon^3\ \cot\varepsilon^2\, p_3,$$

$$\overline{\mathcal{S}}^2 = \cos\varepsilon^3\, p_2 + \frac{\sin\varepsilon^3}{\sin\varepsilon^2}\, p_1 - \sin\varepsilon^3\ \cot\varepsilon^2\, p_3,$$

$$\overline{\mathcal{S}}^3 = p_3, \tag{7.23}$$

and we get

$$\mathcal{S}^3 = p_1 = -\sin\varepsilon^2 \cos\varepsilon^3\ \overline{\mathcal{S}}^1 + \sin\varepsilon^2 \sin\varepsilon^3\, \overline{\mathcal{S}}^2 + \cos\varepsilon^2\, \overline{\mathcal{S}}^3,$$

$$\beta = \left(\varepsilon^1 - \frac{\pi}{2}\right) - \arctan \frac{\sin\varepsilon^2\, (-\sin\varepsilon^2 \cos\varepsilon^3\, \overline{\mathcal{S}}^1 + \sin\varepsilon^2\ \sin\varepsilon^3\, \overline{\mathcal{S}}^2 + \cos\varepsilon^2\, \overline{\mathcal{S}}^3) - \overline{\mathcal{S}}^3}{\sin\varepsilon^2\, (\sin\varepsilon^3\, \overline{\mathcal{S}}^1 + \cos\varepsilon^3\, \overline{\mathcal{S}}^2)}$$

$$\alpha = \arctan \frac{\sin\varepsilon^2 (\sin\varepsilon^3\, \overline{\mathcal{S}}^1 + \cos\varepsilon^3\, \overline{\mathcal{S}}^2)\ \mid \vec{\mathcal{S}} \mid}{\cos\varepsilon^2\ \mid \vec{\mathcal{S}} \mid^2 - (-\sin\varepsilon^2 \cos\varepsilon^3\, \overline{\mathcal{S}}^1 + \sin\varepsilon^2\ \sin\varepsilon^3\, \overline{\mathcal{S}}^2 + \cos\varepsilon^2\, \overline{\mathcal{S}}^3)\, \overline{\mathcal{S}}^3}$$

$$\mid \vec{\mathcal{S}} \mid^2\ = \sum_r \left(\overline{\mathcal{S}}^r\right)^2. \tag{7.24}$$

This chain of transformations can be represented with the table

$$\boxed{\begin{array}{ccc} \mid \vec{\mathcal{S}} \mid & \mathcal{S}^3 & \overline{\mathcal{S}}^3 \\ \alpha & \beta & \gamma \end{array}} \longrightarrow \boxed{\begin{array}{ccc} \overline{\mathcal{S}}^1 & \overline{\mathcal{S}}^2 & \overline{\mathcal{S}}^3 \\ \varepsilon^1 & \varepsilon^2 & \varepsilon^3 \end{array}} \longrightarrow \boxed{\begin{array}{ccc} p_1 & p_2 & p_3 \\ \varepsilon^1 & \varepsilon^2 & \varepsilon^3 \end{array}}, \tag{7.25}$$

and with the following Poisson brackets for the non-canonical variables ε^r, $\overline{\mathcal{S}}^r$ [29] ($\overline{f}$, $\overline{g}$ are functions only of these variables)

$$\{\varepsilon^r, \varepsilon^s\} = 0, \qquad \{\overline{\mathcal{S}}^r, \overline{\mathcal{S}}^s\} = -\epsilon^{rsu}\, \overline{\mathcal{S}}^u,$$

$$\{\varepsilon^r, \overline{\mathcal{S}}_s\} = \check{X}^{(R)\,r}{}_s(\varepsilon^u),$$

$$\{\overline{f}, \overline{g}\} = \check{X}^{(R)\,r}{}_s(\varepsilon^u) \left(\frac{\partial \overline{f}}{\partial \varepsilon^r} \frac{\partial \overline{g}}{\partial \overline{\mathcal{S}}_s} - \frac{\partial \overline{f}}{\partial \overline{\mathcal{S}}_s} \frac{\partial \overline{g}}{\partial \varepsilon^r} \right) - \vec{\mathcal{S}} \cdot \left(\frac{\partial \overline{f}}{\partial \vec{\mathcal{S}}} \times \frac{\partial \overline{g}}{\partial \vec{\mathcal{S}}} \right), \tag{7.26}$$

where $\check{X}^{(R)\,r}{}_s(\varepsilon^u)$ are the components of the right invariant vector fields on the group manifold of $SO(3)$. The components of the dual right invariant one-forms are

$$\Lambda^{(R)\,r}{}_s = \begin{pmatrix} -\sin\varepsilon^2\,\cos\varepsilon^3 & \sin\varepsilon^3 & 0 \\ \sin\varepsilon^2\,\sin\varepsilon^3 & \cos\varepsilon^3 & 0 \\ \cos\varepsilon^2 & 0 & 1 \end{pmatrix} = \left[\check{X}^{(R)\,-1}\right]^r{}_s,$$

if the Euler angles are defined by the convention $R(\varepsilon^r) = R_3(\varepsilon^1)\,R_2(\varepsilon^2)\,R_3(\varepsilon^3)$.

These transformations stress the canonical equivalence between the rotational variables and the canonical phase space of a non relativistic rigid body [32,33]; in other words, the rotational variables are the rigid body-like variables, whereas the *shape* variables describes the *non-rigidity* of the system.

Using the results obtained in the three body case in Refs. [13,14] it is easy to construct the inverse canonical transformation and to express the original relative variables in terms of the rotational and shape variables. In particular it is trivial to observe that by using the rotation $R^r_s(\varepsilon^1, \varepsilon^2, \varepsilon^3)$, for $\underline{n} \neq \underline{u}_1, \underline{u}_2$ we obtain immediately

$$r^r_{\underline{n}} = R^r_s(\varepsilon^1, \varepsilon^2, \varepsilon^3)\,\overline{r}^s_{\underline{n}},$$

$$p^r_{\underline{n}} = R^r_s(\varepsilon^1, \varepsilon^2, \varepsilon^3)\,\overline{p}^s_{\underline{n}}. \tag{7.27}$$

For $\underline{n} = \underline{u}_1, \underline{u}_2$ we can to use the three body results of Refs. [13,14]. In particular we have

$$\begin{aligned} \overline{r}^1_{\underline{u}_i} &= \vec{r}_{\underline{u}_i} \cdot \hat{\chi} = (-)^{i+1} R_i \sqrt{1 - \vec{N}^2}, \\ \overline{r}^2_{\underline{u}_i} &= \vec{r}_{\underline{u}_i} \cdot (\hat{\chi} \times \hat{N}) = 0, \\ \overline{r}^3_{\underline{u}_i} &= \vec{r}_{\underline{u}_i} \cdot \hat{N} = R_i \mid \vec{N} \mid . \end{aligned} \tag{7.28}$$

Then we obtain

$$r^r_{\underline{u}_i} = R^r_s(\varepsilon^1, \varepsilon^2, \varepsilon^3)\,\overline{r}^s_{\underline{u}_i}. \tag{7.29}$$

Finally, if we define

$$\overline{\mathcal{S}}^r_{(12)} = \overline{\mathcal{S}}^r - \sum_{\underline{n} \neq \underline{u}_1, \underline{u}_2} \epsilon^{ruv} \overline{r}^u_{\underline{n}} \overline{p}^s_{\underline{n}}, \tag{7.30}$$

we have

$$\begin{aligned}
\overline{p}^1_{\underline{u}_i} &= (-)^{i+1} \tilde{\Pi}_i \sqrt{1 - \vec{N}^2} + \frac{\mid \vec{N} \mid}{2R_i} \Big[\overline{\mathcal{S}}^2_{(12)} + (-)^{i+1} \xi \sqrt{1 - \vec{N}^2} \Big], \\
\overline{p}^2_{\underline{u}_i} &= (-)^{i+1} \frac{1}{2R_i} \left[-(-)^{i+1} \frac{\overline{\mathcal{S}}^1_{(12)}}{\mid \vec{N} \mid} + \frac{\overline{\mathcal{S}}^3_{(12)}}{\sqrt{1 - \vec{N}^2}} \right], \\
\overline{p}^3_{\underline{u}_i} &= \tilde{\Pi}_i \mid \vec{N} \mid - (-)^{i+1} \frac{\sqrt{1 - \vec{N}^2}}{2R_i} \Big[\overline{\mathcal{S}}^2_{(12)} + (-)^{i+1} \xi \sqrt{1 - \vec{N}^2} \Big],
\end{aligned} \tag{7.31}$$

and then

$$p^r_{\underline{u}_i} = R^r_s(\varepsilon^1, \varepsilon^2, \varepsilon^3)\, \overline{p}^s_{\underline{u}_i}. \tag{7.32}$$

Therefore we get

$$\begin{aligned}
\Sigma^r(\tau, \vec{\sigma}_o) &= R^r{}_s(\varepsilon^u) \sum_{\underline{n}} \overline{r}^s_{\underline{n}}(\tau) \Big[\Gamma^\Sigma_{\underline{n}}(\vec{\sigma}_o) - h_{\underline{n}} \Big] \rightarrow_{\tau \rightarrow 0} \sigma^r_o, \\
K^r(\tau, \vec{\sigma}_o) &= R^r{}_s(\varepsilon^u) \sum_{\underline{n}} \overline{p}^s_{\underline{n}}(\tau)\, \Gamma^K_{\underline{n}}(\vec{\sigma}_o).
\end{aligned} \tag{7.33}$$

VIII. THE INVARIANT MASS AND THE EQUATIONS OF MOTION.

As we have seen, the invariant mass (Dixon's mass monopole as we shall see in the next Section) $\mathcal{M} = \int d^3\sigma_o\, \Delta(\tau, \vec{\sigma}_o)$ is the Hamiltonian on the Wigner hyper-planes in the gauge $T \approx \tau$ and $\vec{\mathcal{Q}} \approx 0$, where $\vec{\Sigma}(\tau, \vec{\sigma}_o)$ and $\vec{K}(\tau, \vec{\sigma}_o)$ are given by Eqs.(7.33). Therefore we have $\frac{d}{d\tau}\, \mathcal{M} = 0$, but it can be shown that $\frac{\partial}{\partial \tau}\, \Delta(\tau, \vec{\sigma}_o) \neq 0$. Moreover, since $h_{\underline{n}}$ in Eqs.(7.33) depends on both configuration and momentum shape variables, it can be shown that we have $\frac{d}{d\tau}\, h_{\underline{n}} \neq 0$, namely

$$\frac{\partial \Sigma^r(\tau, \vec{\sigma}_o)}{\partial \tau} = \Big(\vec{\omega}(\tau) \times \vec{\Sigma}(\tau, \vec{\sigma}_o)\Big)^r + \\ + R^r{}_s(\varepsilon^u(\tau)) \sum_{\underline{n}} \left(\frac{d\overline{r}^s_{\underline{n}}(\tau)}{d\tau} \Big[\Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o) - h_{\underline{n}}(\tau)\Big] - \overline{r}^s_{\underline{n}}(\tau)\, \frac{dh_{\underline{n}}(\tau)}{d\tau}\right), \tag{8.1}$$

with the body frame components of the angular velocity being $\overline{\omega}^r(\varepsilon^u(\tau), \dot{\varepsilon}^u(\tau)) = -\frac{1}{2}\, \epsilon^{ruv} \Big[R^T\, \dot{R}\Big]^{uv} (\varepsilon^u(\tau), \dot{\varepsilon}^u(\tau))$.

As already said, for every equation of state the mass density $\Delta(\tau, \vec{\sigma}_o)$ is a suitable function of $n_o(\vec{\sigma}_o)$, $\vec{K}^2(\tau, \vec{\sigma}_o)/n_o^2(\vec{\sigma}_o)$ and $n_o(\vec{\sigma}_o)/\det\Big(\frac{\partial \Sigma}{\partial \sigma_o}\Big)$. While the last term (absent only in the case of dust, because $p = 0$) is determined by Eq.(6.31), from Eq.(6.30) we get the following expression of the second term

$$\begin{aligned}
\frac{\vec{K}^2(\tau, \vec{\sigma}_o)}{n_o^2(\vec{\sigma}_o)} &= \frac{1}{n_o^2(\vec{\sigma}_o)} \sum_{\underline{n}_1, \underline{n}_2} \Gamma^K_{\underline{n}_1}(\vec{\sigma}_o)\, \Gamma^K_{\underline{n}_2}(\vec{\sigma}_o)\, \vec{p}_{\underline{n}_1}(\tau) \cdot \vec{p}_{\underline{n}_2}(\tau) = \\
&= \frac{1}{n_o^2(\vec{\sigma}_o)} \Big[\sum_{i=1}^2 \Big(\Gamma^K_{\underline{u}_i}(\vec{\sigma}_o)\Big)^2\, \vec{p}^{\,2}_{\underline{u}_i}(\tau) + 2\, \Gamma^K_{\underline{u}_1}(\vec{\sigma}_o)\, \Gamma^K_{\underline{u}_2}(\vec{\sigma}_o)\, \vec{p}_{\underline{u}_1}(\tau) \cdot \vec{p}_{\underline{u}_2}(\tau) + \\
&+ 2 \sum_{i=1}^2 \Gamma^K_{\underline{u}_i}(\vec{\sigma}_o) \sum_{\underline{n} \neq \underline{u}_1, \underline{u}_2} \Gamma^K_{\underline{n}}(\vec{\sigma}_o)\, \vec{p}_{\underline{u}_i}(\tau) \cdot \vec{p}_{\underline{n}}(\tau) + \\
&+ \sum_{\underline{n}_1 \neq \underline{u}_1, \underline{u}_2} \sum_{\underline{n}_2 \neq \underline{u}_1, \underline{u}_2} \Gamma^K_{\underline{n}_1}(\vec{\sigma}_o)\, \Gamma^K_{\underline{n}_2}(\vec{\sigma}_o)\, \vec{p}_{\underline{n}_1}(\tau) \cdot \vec{p}_{\underline{n}_2}(\tau)\Big].
\end{aligned} \tag{8.2}$$

By using Eqs. (7.10) and (7.31) this term can be expressed in the non-canonical basis ε^r, $\overline{\mathcal{S}}^r$, $\mid \vec{N} \mid$, ξ, R_1, $\tilde{\Pi}_1$, R_2, $\tilde{\Pi}_2$, $\overline{r}^s_{\underline{n} \neq \underline{u}_1, \underline{u}_2}$, $\overline{p}^s_{\underline{n} \neq \underline{u}_1, \underline{u}_2}$. The result is that the invariant mass density

i) is independent from the Euler angles ε^r;

ii) contains terms bilinear and linear in the body frame components $\overline{\mathcal{S}}^r$ of the spin.

However, $\Delta(\tau, \vec{\sigma}_o)$ is a complicated function of these three terms. The simplest expression is obtained in the case of dust, where $\Delta(\tau, \vec{\sigma}_o) = \sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 + \vec{K}^2(\tau, \vec{\sigma}_o)}$. Since the body frame components of the angular velocity are defined [13,14,22] as [12]

$$\overline{\omega}_r(\varepsilon^s) = \frac{\partial \mathcal{M}(\tau)}{\partial \overline{\mathcal{S}}^r} = F_{rs}(\overline{\mathcal{S}}^u, q^\mu, p_\mu)\, \overline{\mathcal{S}}^s + G_r(\overline{\mathcal{S}}^u, q^\mu, p_\mu), \tag{8.3}$$

it turns out that there is no linear relation between the spin and the angular velocity like in the non-relativistic rigid body (this property is true also for non-relativistic non-rigid bodies [13]).

We can write the Hamilton equations for the orientational variables ε^r, $\overline{\mathcal{S}}^r$ and for the shape variables. From them we can deduce the equations of motion for the orientational variables α, $\overline{\mathcal{S}}^3$, γ in Eqs.(7.10) (the other three $|\, \vec{\mathcal{S}}\, |$, $\mathcal{S}^3$, β are Noether constants of motion). These three variables are not constant of motion for deformable bodies: they are coupled to the shape variables and *describe how the dynamical body frame rotates when the body changes its shape*. In particular α, being conjugate to the constant of motion $|\, \vec{\mathcal{S}}\, |$, is an ignorable variable ($\mathcal{M}$, expressed in the canonical basis (7.10), does not depend on it).

Three types of configuration for the motion of the fluid are interesting:

i) *Pure rotational motion* - It is defined by constant shape configuration variables $\dot{q}^\mu = 0$. With this condition the Hamilton equation $\dot{q}^\mu \circeq \{q^\mu, \mathcal{M}\}$ become a system of algebraic equations for the shape momenta $p^{(o)}_\mu = p_\mu|_{\dot{q}=0}$. Even if we cannot find the explicit solution, its form is of the type $p^{(o)}_\mu = \sum_r \overline{\mathcal{S}}^r\, \mathcal{C}^r_\mu(\overline{\mathcal{S}}^u, q^\nu)$. The purely rotational Hamiltonian is $\mathcal{M}^{(rot)} = \mathcal{M}|_{p_\mu = p^{(o)}_\mu}$. Therefore for the dust invariant mass density we get

$$\Delta^{(rot)}(\tau, \vec{\sigma}_o) = \sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 + \sum_{rs} A^{rs}(\vec{\sigma}_o, q^\mu(\tau), \overline{\mathcal{S}}^u)\, \overline{\mathcal{S}}^r\, \overline{\mathcal{S}}^s}. \tag{8.4}$$

However Eqs.(8.1) show that the *generalized Eulerian coordinates* (i.e. the flux lines in adapted coordinates) *do not perform a rigid motion*:

$$\frac{\partial \Sigma^r(\tau, \vec{\sigma}_o)}{\partial \tau}|_{\dot{q}=0} = \Big(\vec{\omega}(\tau) \times \vec{\Sigma}(\tau, \vec{\sigma}_o)\Big)^r - R^r{}_s(\varepsilon^u(\tau)) \sum_{\underline{n}} \overline{r}^s_{\underline{n}}(\tau)\, \frac{\partial h_{\underline{n}}(\tau)}{\partial p^{(o)}_\nu}\, \dot{p}^{(o)}_\mu(\tau) \neq$$

$$\neq \Big(\vec{\omega}(\tau) \times \vec{\Sigma}(\tau, \vec{\sigma}_o)\Big)^r. \tag{8.5}$$

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[12]From now on we shall use the following notations: i) q^μ, p_μ will denote all the canonically conjugate shape variables $|\vec{N}|$, ξ, R_1, $\tilde{\Pi}_1$, R_2, $\tilde{\Pi}_2$, $\overline{r}^s_{\underline{n}\neq\underline{u}_1,\underline{u}_2}$, $\overline{p}^s_{\underline{n}\neq\underline{u}_1,\underline{u}_2}$; ii) q^α, p_α will denote all the canonically conjugate shape variables $\overline{r}^s_{\underline{n}\neq\underline{u}_1,\underline{u}_2}$, $\overline{p}^s_{\underline{n}\neq\underline{u}_1,\underline{u}_2}$ not including the first three pairs connected with the choice of the body frame axes.

ii) *Pure vibrational motion* - It is defined by the vanishing of the angular velocity, so that Eq.(8.3) determine the body frame components of the spin in terms of the shape variables: $\overline{\mathcal{S}}^r|_{\vec{\omega}=0} = \overline{\mathcal{S}}^r_{(o)}(q^\mu, p_\mu) \neq 0$. By putting this expression in $\mathcal{M}$ gives a purely vibrational Hamiltonian $\mathcal{M}^{(vib)}$.

iii) *Small shape momenta* - We can study configurations in which the shape momenta are very small. We can define the following two approximations (we use the dust to illustrate them):

a) $p_\mu \sim 0$ - The dust invariant mass density becomes

$$\Delta(\tau, \vec{\sigma}_o) \sim \sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 + \sum_{rs} C^{rs}(\vec{\sigma}_o, q^\mu)\, \overline{\mathcal{S}}^r\, \overline{\mathcal{S}}^s}. \tag{8.6}$$

If we make a Taylor expansion around the canonical center of mass $\vec{\mathcal{Q}} = 0$ of the function [see also Eq. (9.2)]

$$\widehat{T}^{\tau\tau}(\tau, \vec{\sigma}_o) = \det{}^{-1}\left(\frac{\partial\Sigma}{\partial\sigma_o}\right) \Delta(\tau, \vec{\sigma}_o) \tag{8.7}$$

and if we define

$$D_o(\tau) = \det{}^{-1}\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)_{\vec{\sigma}_o=0} \tag{8.8}$$

we can define the following new Δ-multipolar expansion of the invariant mass ($V(\tau)$ is the volume of the fluid)

$$\begin{aligned} \mathcal{M} &\sim V(\tau)\, D_o(\tau)\, \sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 + \sum_{rs} C^{rs}(\vec{0}, q^\mu)\, \overline{\mathcal{S}}^r\, \overline{\mathcal{S}}^s} + \\ &+ \sum_{n=1}^{\infty} \sum_{r_1..r_n} \widehat{T}_{r_1..r_n}(q^\mu, \overline{\mathcal{S}}^u) \int_{V(0)} d^3\sigma_o\, \det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right) \sigma_o^{r_1}...\sigma_o^{r_n}, \end{aligned} \tag{8.9}$$

where

$$\widehat{T}_{r_1..r_n} = \left[\frac{\partial}{\partial\sigma_o^{r_1}...\partial\sigma_o^{r_n}} \widehat{T}^{\tau\tau}(\tau, \vec{\sigma}_o)\right]_{\vec{\sigma}_o=0}. \tag{8.10}$$

The first term in this multipolar expansion is just the *relativistic rotator* defined in Eq.(5.12) of Ref. [27] and $C^{rs}(\vec{0}, q^\mu)$ plays the role of the inverse of the tensor of inertia.

b) Only $p_\alpha \sim 0$ - If we denote U^i the 3 shape variables $|\vec{N}|$, R_1, R_2 and V_i their conjugate momenta ξ, $\tilde{\Pi}_1$, $\tilde{\Pi}_2$, the dust invariant mass density becomes

$$\Delta(\tau, \vec{\sigma}_o) \sim \sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 + \sum_{rs} C^{rs}(\vec{\sigma}_o, q^\alpha, U^i)\, \overline{\mathcal{S}}^r\, \overline{\mathcal{S}}^s + \sum_r\, B^r(\vec{\sigma}_o, q^\alpha, U^i, V_i)\, \overline{\mathcal{S}}^r + H_{(3)}(U^i, V_i)},$$

where $H_{(3)}$ is the non-relativistic vibrational (namely only function of the shape variables) Hamiltonian for the 3-body problem [13]. If we repeat the previous Taylor expansion (Δ-multipolar expansion), the first term will be

$$V(\tau)\, D_o(\tau) \sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 + \sum_{rs} C^{rs}(\vec{0}, q^\alpha, U^i)\, \overline{\mathcal{S}}^r\, \overline{\mathcal{S}}^s + \sum_r\, B^r(\vec{0}, q^\alpha, U^i, V_i)\, \overline{\mathcal{S}}^r + H_{(3)}(U^i, V_i)}$$

It corresponds to a generalized rotator interacting with a 3-body problem parametrized by the two dipoles R_1, R_2, originally along two of the chosen body frame axes, and by the angle between them, described by $|\vec{N}|$.

IX. DIXON'S MULTIPOLES.

In this Section, following the general treatment given in Ref. [17], we shall give Dixon's multipoles [16,17] for the energy-momentum tensor of the perfect fluid in the rest-frame instant form on the Wigner hyper-planes in the gauge $T \approx \tau$ and $\vec{\mathcal{Q}} \approx 0$.

On the Wigner hyper-planes the energy-momentum tensor (2.22) is rewritten in the form

$$T^{AB}(\tau, \vec{\sigma}) = \int d^3\sigma_o \, \det\left(\frac{\partial \Sigma}{\partial \sigma_o}\right) \, \delta^3(\vec{\sigma} - \vec{\Sigma}(\tau, \vec{\sigma}_o)) \, \hat{T}^{AB}(\tau, \vec{\sigma}_o), \tag{9.1}$$

with ($\widehat{U}^A$ and $\mathcal{R}$ are defined in Eqs. (3.16) and (3.10), respectively)

$$\widehat{T}^{AB}(\tau, \vec{\sigma}_o) = \left[\left(\frac{\partial \rho}{\partial \hat{n}} \, \hat{n}\right) \widehat{U}^A \, \widehat{U}^B + \left(\rho - \frac{\partial \rho}{\partial \hat{n}} \, \hat{n}\right) g^{AB}\right](\tau, \vec{\sigma}_o),$$

$$\widehat{U}^A(\tau, \vec{\sigma}_o) = \frac{1}{\mathcal{R}(\tau, \vec{\sigma}_o)} \Big(1; \, \frac{\partial \vec{\Sigma}(\tau, \vec{\sigma}_o)}{\partial \tau}\Big),$$

$$\widehat{T}^{\tau\tau}(\tau, \vec{\sigma}_o) = \Delta(\tau, \vec{\sigma}_o) \det{}^{-1}\left(\frac{\partial \Sigma}{\partial \sigma_o}\right),$$

$$\widehat{T}^{\tau r}(\tau, \vec{\sigma}_o) = K^r(\tau, \vec{\sigma}_o) \det{}^{-1}\left(\frac{\partial \Sigma}{\partial \sigma_o}\right),$$

$$\widehat{T}^{rs}(\tau, \vec{\sigma}_o) = \frac{\det^{-1}\left(\frac{\partial \Sigma}{\partial \sigma_o}\right) \, K^r(\tau, \vec{\sigma}_o) \, K^s(\tau, \vec{\sigma}_o)}{\sqrt{\Big(n_o(\vec{\sigma}_o) \, \frac{\partial \rho}{\partial X}(X, \hat{s}_o)\Big)^2 + \vec{K}^2(\tau, \vec{\sigma}_o)}} + \hat{p}(X, \hat{s}_o) \, \delta^{rs}. \tag{9.2}$$

Let us consider a world-line $w^\mu(\tau) = x^\mu(\tau) + \epsilon^\mu_r(p) \, \zeta^r(\tau)$, where $x^\mu(\tau)$ is the centroid, origin of the 3-coordinates on Wigner hyper-planes. Dixon's multipoles of the energy-momentum tensor with respect to this world-line are defined as

$$t^{\mu_1 \ldots \mu_n \mu\nu}(w(\tau)) = \int d^3\sigma \, \left(z^{\mu_1}(\tau, \vec{\sigma}) - w^{\mu_1}(\tau)\right) \ldots \left(z^{\mu_n}(\tau, \vec{\sigma}) - w^{\mu_n}(\tau)\right) \, T^{\mu\nu}(z(\tau, \vec{\sigma})) =$$

$$= \epsilon^{\mu_1}_{r_1}(p) \ldots \epsilon^{\mu_n}_{r_n}(p) \, \epsilon^\mu_A(p) \, \epsilon^\nu_B(p) \, q^{r_1 .. r_n AB}(\vec{\zeta}(\tau)). \tag{9.3}$$

The rest-frame instant form multipoles are

$$q^{r_1 .. r_n AB}(\vec{\zeta}(\tau)) = \int d^3\sigma \, \left(\sigma^{r_1} - \zeta^{r_1}(\tau)\right) \ldots \left(\sigma^{r_n} - \zeta^{r_n}(\tau)\right) \, T^{AB}(\tau, \vec{\sigma}) =$$

$$= \int d^3\sigma_o \, \left(\Sigma^{r_1}(\tau, \vec{\sigma}_o) - \zeta^{r_1}(\tau)\right) \ldots \left(\Sigma^{r_1}(\tau, \vec{\sigma}_o) - \zeta^{r_1}(\tau)\right) \det\left(\frac{\partial \Sigma}{\partial \sigma_o}\right) \, \hat{T}^{AB}(\tau, \vec{\sigma}_o). \tag{9.4}$$

The *monopoles* are $q^{AB}(\vec{\zeta}(\tau))$ with the *mass monopole* $q^{\tau\tau}(\vec{\zeta}(\tau)) = \mathcal{M}$ and the vanishing (due to the rest frame condition) *momentum monopole* $q^{\tau r}(\vec{\zeta}(\tau)) = \mathcal{P}^r \approx 0$. Then there are the *stress tensor monopole* $q^{rs}(\vec{\zeta}(\tau))$ and the trace $q^A{}_A(\vec{\zeta}(\tau))$.

The *dipoles* are $q^{rAB}(\vec{\zeta}(\tau))$. If we ask for the vanishing of the *mass dipole*

$$q^{r\tau\tau}(\vec{\zeta}(\tau)) = \int d^3\sigma_o\ [\Sigma^r(\tau,\vec{\sigma}_o) - \zeta^r(\tau)]\ \Delta(\tau,\vec{\sigma}_o),$$

we find

$$q^{r\tau\tau}(\vec{\zeta}(\tau)) = 0 \quad \Rightarrow \vec{\zeta}(\tau) = \vec{\mathcal{R}} \approx \vec{\mathcal{Q}} \approx 0. \tag{9.5}$$

This means that the vanishing of the mass dipole identifies the world-line of the internal Møller center of energy, namely the centroid $x^\mu(\tau)$, which, in the rest-frame instant form in the gauge $\vec{\mathcal{Q}} \approx 0$, is also both Tulczyjew and Pirani centroid, as shown in Ref. [17] in the case of particles.

Therefore the multipoles with respect to the center of energy are

$$\begin{aligned}
&q^{r_1..r_n AB}(\tau) = \\
&= \int d^3\sigma_o\ (\Sigma^{r_1}(\tau,\vec{\sigma}_o) - \mathcal{R}^{r_1}(\tau))\ ...\ (\Sigma^{r_1}(\tau,\vec{\sigma}_o) - \mathcal{R}^{r_1}(\tau))\ \det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\ \hat{T}^{AB}(\tau,\vec{\sigma}_o) = \\
&= \sum_{\underline{m}_1,..,\underline{m}_n} r^{r_1}_{\underline{m}_1}(\tau)..r^{r_n}_{\underline{m}_n}(\tau) \int d^3\sigma_o\ \Big(\Gamma^\Sigma_{\underline{m}_1}(\tau,\vec{\sigma}_o) - h_{\underline{m}_1}(\tau)\Big)\ ...\ \Big(\Gamma^\Sigma_{\underline{m}_n}(\tau,\vec{\sigma}_o) - h_{\underline{m}_n}(\tau)\Big) \times \\
&\times \det\left(\sum_{\underline{n}} \frac{\partial\,\Gamma^\Sigma_{\underline{n}}(\vec{\sigma}_o)}{\partial\sigma^s_o}\, r^r_{\underline{n}}(\tau)\right)\ \hat{T}^{AB}(\tau,\vec{\sigma}_o).
\end{aligned} \tag{9.6}$$

The last lines give the expression of Dixon's multipoles in terms of the relative variables, once the energy momentum tensor (9.2) is rewritten in terms of them. Then we could re-express the multipoles in terms of the orientational and shape variables.

Then the *momentum dipole* takes the following form

$$\begin{aligned}
q^{r\,s\tau}(\tau) &= \int d^3\sigma_o\ \Big(\Sigma^r(\tau,\vec{\sigma}_o) - \mathcal{R}^r(\tau)\Big)\ K^s(\tau,\vec{\sigma}_o) = \\
&= \int d^3\sigma_o\ \sum_{\underline{m}} r^r_{\underline{m}}(\tau)\ \Big(\Gamma^\Sigma_{\underline{m}}(\tau,\vec{\sigma}_o) - h_{\underline{m}}(\tau)\Big)\ K^s(\tau,\vec{\sigma}_o) \approx \\
&\approx \sum_{\underline{n}} r^r_{\underline{n}}(\tau)\, p^s_{\underline{n}}(\tau),
\end{aligned} \tag{9.7}$$

and the angular momentum with respect to the internal center of energy, coinciding with the internal spin $\vec{\mathcal{S}}$, is

$$\mathcal{S}^u = \frac{1}{2}\epsilon^{urs}\Big[q^{r\,s\tau}(\tau) - q^{s\,r\tau}(\tau)\Big] \approx$$

$$\approx \frac{1}{2}\epsilon^{urs}\sum_{\underline{n}}\Big(r^r_{\underline{n}}(\tau)\,p^s_{\underline{n}}(\tau) - r^s_{\underline{n}}(\tau)\,p^r_{\underline{n}}(\tau)\Big). \tag{9.8}$$

As shown in Ref. [17] the *quadrupoles* $q^{r_1r_2\,AB}(\tau)$ allow to introduce two definitions of *barycentric tensor of inertia*:

i) Dixon's one using the mass quadrupole, $I^{r_1r_2}_{dixon}(\tau) = \delta^{r_1r_2}\,\sum_u q^{uu\tau\tau}(\tau) - q^{r_1r_2\tau\tau}(\tau)$;

ii) Thorne's one, $I^{r_1r_2}_{thorne}(\tau) = \delta^{r_1r_2}\,\sum_u q^{uuA}{}_A(\tau) - q^{r_1r_2A}{}_A(\tau)$;

both definitions give the standard tensor of inertia in the non-relativistic limit, since their difference is at the post-Newtonian order.

In Ref. [17] there is the study of the non-relativistic limit of Dixon's multipoles by means of the Gartenhaus-Schwartz transformation, which are then compared with the non-relativistic multipoles defined in Appendix A of that paper.

X. CONCLUSIONS.

We have developed a new Lagrangian and Hamiltonian formulation of perfect fluids in terms of generalized Eulerian coordinates $\vec{\Sigma}(\tau, \vec{\sigma}_o)$, which not only produces less complicated invariant masses, but also allows to define dynamical body frames, spin frames, orientational and shape variables by a natural extension of the techniques developed for N-body systems. On the contrary, it is too difficult to determine these quantities in the description based on the Lagrangian comoving coordinates. Since the Lagrangian contains the fluid density at the initial time, the choice of the two first axes of the dynamical body frame can be adapted to the initial form of the fluid. Then the dynamical body frame will evolve in time according to the changes in the form of the fluid according to the initial data and to the equation of state.

We have also evaluated Dixon's multipoles in the rest-frame instant form. While they allow a description of the mean motion of the extended body, the fluid in this case, the orientational and shape variables give a complete information about the real motion with its changes of shape, moreover adapted to all the generic Noether constants of motion. The invariant mass of the fluid, i.e. the Hamiltonian governing the real motion, can in turn be expressed with a Δ-multipolar expansion, as shown in Section VIII, which may be more useful than Dixon's multipoles when only a finite number of shape variables is relevant (the others may be treated as perturbations).

We hope that this description of the fluid as an extended deformable relativistic body will help to treat with numerical simulations and/or approximations any type of system from droplet models of the proton, to heavy ion fireballs, plasmas and rotating stars. Regarding the simulation of stars we still need to make the coupling to tetrad gravity and then to go to a completely fixed 3-orthogonal gauge with the technique developed in Ref. [20] in absence of matter. This will give a new starting point for Hamiltonian numerical gravity and for the simulation of the properties of rotating stars. In particular in the case of incompressible fluids it will be interesting to try to recover the ellipsoidal equilibrium configurations in the non-relativistic limit. In the meantime in Ref. [22] it will be shown that the non-relativistic limit of the formalism of this paper allows to describe such configuration after the addition of the Newton gravitational potential.

APPENDIX A: DEFINITIONS AND RELATIONS ON THE SPACE-LIKE HYPER-SURFACE

The *vierbeins* $z^\mu_{\check A}(\tau,\vec\sigma) = \frac{\partial z^\mu(\tau,\vec\sigma)}{\partial \sigma^{\check A}}$ satisfy the *Gauss-Codazzi-Weingarten* integrability relation

$$\frac{\partial}{\partial \sigma^{\check B}} z^\mu_{\check A}(\tau,\vec\sigma) - \frac{\partial}{\partial \sigma^{\check A}} z^\mu_{\check B}(\tau,\vec\sigma) = 0. \tag{A1}$$

By construction the three 4-vectors $z^\mu_{\check r}(\tau,\vec\sigma)$ ($\check r = 1,2,3$) define a *non-orthonormal* basis for the tangent space to hyper-surface $\Sigma(\tau)$; then it is possible to define the induced metric on the hyper-surface

$$g_{\check r\check s}(\tau,\vec\sigma) = z^\mu_{\check r}(\tau,\vec\sigma)\eta_{\mu\nu} z^\nu_{\check s}(\tau,\vec\sigma), \tag{A2}$$

and the *normal unit vector*

$$l^\mu(\tau,\vec\sigma) = \frac{1}{\sqrt{\gamma(\tau,\vec\sigma)}} \epsilon^{\mu\alpha\beta\gamma} z_{1\alpha}(\tau,\vec\sigma) z_{2\beta}(\tau,\vec\sigma) z_{3\gamma}(\tau,\vec\sigma), \tag{A3}$$

where

$$\gamma(\tau,\vec\sigma) = -\det(g_{\check r\check s}(\tau,\vec\sigma)), \tag{A4}$$

so that

$$l^\mu(\tau,\vec\sigma) l_\mu(\tau,\vec\sigma) = 1. \tag{A5}$$

Equally if we define

$$g(\tau,\vec\sigma) = -\det(g_{\check A\check B}(\tau,\vec\sigma)), \tag{A6}$$

we have

$$l_\mu(\tau,\vec\sigma) z^\mu_\tau(\tau,\vec\sigma) = \sqrt{\frac{g(\tau,\vec\sigma)}{\gamma(\tau,\vec\sigma)}}. \tag{A7}$$

We can define the inverse vierbeins $z^{\check A}_\mu(\tau,\vec\sigma)$ such that

$$\begin{aligned} z^{\check A}_\mu(\tau,\vec\sigma) z^\mu_{\check B}(\tau,\vec\sigma) &= \delta^{\check A}_{\check B}, \\ z^\mu_{\check A}(\tau,\vec\sigma) z^{\check A}_\nu(\tau,\vec\sigma) &= \eta^\mu_\nu. \end{aligned} \tag{A8}$$

Then the inverse metric $g^{\check{A}\check{B}}(\tau,\vec{\sigma})$, such that

$$g^{\check{A}\check{B}}(\tau,\vec{\sigma})g_{\check{B}\check{C}}(\tau,\vec{\sigma}) = \delta^{\check{A}}_{\check{C}}, \tag{A9}$$

is defined by

$$g^{\check{A}\check{B}}(\tau,\vec{\sigma}) = z^{\check{A}}_{\mu}(\tau,\vec{\sigma})\eta^{\mu\nu}z^{\check{B}}_{\nu}(\tau,\vec{\sigma}). \tag{A10}$$

We have also

$$\eta^{\mu\nu} = z^{\mu}_{\check{A}}(\tau,\vec{\sigma})g^{\check{A}\check{B}}(\tau,\vec{\sigma})z^{\nu}_{\check{B}}(\tau,\vec{\sigma}). \tag{A11}$$

It is useful to consider the inverse 3-dimensional metric $\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})$ such that

$$\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})g_{\check{r}\check{s}}(\tau,\vec{\sigma}) = \delta^{\check{r}}_{\check{s}}. \tag{A12}$$

The following relation holds

$$\eta^{\mu\nu} = l^{\mu}(\tau,\vec{\sigma})l^{\nu}(\tau,\vec{\sigma}) + \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})z^{\mu}_{\check{r}}(\tau,\vec{\sigma})z^{\nu}_{\check{s}}(\tau,\vec{\sigma}). \tag{A13}$$

From this expression we can find

$$z^{\mu}_{\tau}(\tau,\vec{\sigma}) = \sqrt{\frac{g(\tau,\vec{\sigma})}{\gamma(\tau,\vec{\sigma})}}l^{\mu}(\tau,\vec{\sigma}) + g_{\tau\check{r}}(\tau,\vec{\sigma})\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})z^{\mu}_{\check{s}}(\tau,\vec{\sigma}). \tag{A14}$$

If we define the *lapse* function

$$N(\tau,\vec{\sigma}) = \sqrt{\frac{g(\tau,\vec{\sigma})}{\gamma(\tau,\vec{\sigma})}}, \tag{A15}$$

and the *shift* function

$$N^{\check{r}}(\tau,\vec{\sigma}) = g_{\tau\check{u}}(\tau,\vec{\sigma})\gamma^{\check{u}\check{r}}(\tau,\vec{\sigma}), \tag{A16}$$

we can write the following expression for the inverse metric

$$\begin{aligned}
g^{\tau\tau}(\tau,\vec{\sigma}) &= \frac{1}{N^2(\tau,\vec{\sigma})},\\
g^{\check{r}\tau}(\tau,\vec{\sigma}) &= -\frac{1}{N^2(\tau,\vec{\sigma})}N^{\check{r}}(\tau,\vec{\sigma}),\\
g^{\check{r}\check{s}}(\tau,\vec{\sigma}) &= \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma}) + \frac{1}{N(\tau,\vec{\sigma})}N^{\check{r}}(\tau,\vec{\sigma})N^{\check{s}}(\tau,\vec{\sigma}).
\end{aligned} \tag{A17}$$

and we have

$$z^\mu_\tau(\tau,\vec{\sigma}) = N(\tau,\vec{\sigma}) l^\mu(\tau,\vec{\sigma}) + N^{\check{s}}(\tau,\vec{\sigma}) z^\mu_{\check{s}}(\tau,\vec{\sigma}). \tag{A18}$$

Moreover from the definition

$$g_{\tau\tau}(\tau,\vec{\sigma}) = z^\mu_\tau(\tau,\vec{\sigma}) \eta_{\mu\nu} z^\nu_\tau(\tau,\vec{\sigma}), \tag{A19}$$

we have

$$g_{\tau\tau}(\tau,\vec{\sigma}) = N^2(\tau,\vec{\sigma}) + N^{\check{r}}(\tau,\vec{\sigma}) g_{\tau\check{r}}(\tau,\vec{\sigma}). \tag{A20}$$

APPENDIX B: PERFECT FLUIDS ADMITTING A CLOSED FORM OF THE INVARIANT MASS.

Let us rewrite Eq.(4.8) in the following form

$$\left(\frac{\partial\rho(X,s)}{\partial X}\right)^2(\tau,\vec{\sigma}_o)\,\Big[X^2-B^2\Big](\tau,\vec{\sigma}_o)=A(\tau,\vec{\sigma}_o)\,X^2(\tau,\vec{\sigma}_o),$$

$$B(\tau,\vec{\sigma}_o)=\frac{n_o(\vec{\sigma}_o)}{\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\,\sqrt{\gamma(\Sigma)}},$$

$$A(\tau,\vec{\sigma}_o)=\frac{\gamma^{\check{r}\check{s}}(\Sigma)K_{\check{r}}(\tau,\vec{\sigma}_o)K_{\check{s}}(\tau,\vec{\sigma}_o)}{n_o^2(\vec{\sigma}_o)}. \tag{B1}$$

For each equation of state $\rho=\rho(\hat{n},\hat{s}_o)$ with $\hat{n}=X$ [13] the solution of this equation allows to get the explicit phase space form of the constraint $\mathcal{H}_\perp(\tau,\vec{\sigma})\approx 0$ in Eqs.(4.9). Referring to Section 5 of Ref. [6] for the determination of the equations of state $\rho=\rho(\hat{n},\hat{s}_o)$, in this Section we will show the few cases in which the solution for $X(\tau,\vec{\sigma}_o)$ can be obtained in closed form.

1) As shown in Ref. [6], for the *dust* we have $p=0$ and $\rho(\hat{n})=\mu\,\hat{n}$ with $\mu=const$. By using Eqs. (3.10) and (3.11), the action (3.15) becomes

$$S=-\mu\int d\tau d^3\sigma_o\,n_o(\vec{\sigma}_o)\,\mathcal{R}(\tau,\vec{\sigma}_o),$$

$$\mathcal{R}(\tau,\vec{\sigma}_o)=\sqrt{\Big(g_{\tau\tau}(\tau,\vec{\Sigma})+2\,g_{\tau\check{r}}(\tau,\vec{\Sigma})\,\frac{\partial\Sigma^{\check{r}}}{\partial\tau}+g_{\check{r}\check{s}}(\tau,\vec{\Sigma})\,\frac{\partial\Sigma^{\check{r}}}{\partial\tau}\,\frac{\partial\Sigma^{\check{s}}}{\partial\tau}\Big)(\tau,\vec{\sigma}_o)}. \tag{B2}$$

Let us remark that with the positions $n_o(\vec{\sigma}_o)=\sum_{i=1}^N\,\delta^3(\vec{\sigma}_o-\vec{\eta}_i(0))$ and $\vec{\Sigma}(\tau,\vec{\eta}_i(0))=\vec{\eta}_i(\tau)$ [so that consistently $\vec{\Sigma}(0,\vec{\eta}_i(0))=\vec{\eta}_i(0)$] we get the action of N free particles of equal mass μ:

$$S=-\mu\sum_{i=1}^N\int d\tau\,\sqrt{g_{\tau\tau}(\tau,\vec{\eta}_i(\tau))+2\,g_{\tau r}(\tau,\vec{\eta}_i(\tau))\,\dot{\eta}_i^r(\tau)+g_{rs}(\tau,\vec{\eta}_i(\tau))\,\dot{\eta}_i^r(\tau)\,\dot{\eta}_i^s(\tau)}. \tag{B3}$$

Since we have $\rho(X)-\frac{\partial\rho(X)}{\partial X}\,X=0$, Eq.(4.8) has the solution

[13] In Ref. [6] $X=\sqrt{\gamma}\,n$ was used as unknown.

$$X(\tau,\vec{\sigma}_o) = \frac{\mu\, n_o^2(\vec{\sigma}_o)}{\sqrt{\gamma(\tau,\vec{\sigma}_o)}\, \det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)\, \sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 - \gamma^{rs}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))\, K_r(\tau,\vec{\sigma}_o)\, K_s(\tau,\vec{\sigma}_o)}} \tag{B4}$$

and the second of Eqs.(4.9) becomes

$$\begin{aligned}
&\mathcal{H}_\perp(\tau,\vec{\sigma}) = \rho_\mu(\tau,\vec{\sigma})\, l^\mu(\tau,\vec{\sigma}) + \\
&- \int d^3\sigma_o\, \delta^3(\vec{\sigma} - \vec{\Sigma}(\tau,\vec{\sigma}_o)) \sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 - \gamma^{rs}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))\, K_r(\tau,\vec{\sigma}_o)\, K_s(\tau,\vec{\sigma}_o)}.
\end{aligned} \tag{B5}$$

On space-like hyper-planes and on Wigner hyper-planes the invariant mass and the internal boost of Eqs.(4.27), (5.21), (5.23) become

$$\begin{aligned}
\mathcal{M} &= \int d^3\sigma_o\, \sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 + \vec{K}^2(\tau,\vec{\sigma}_o)},\\
\vec{\mathcal{K}} &= - \int d^3\sigma_o\, \vec{\Sigma}(\tau,\vec{\sigma}_o)\, \sqrt{[\mu\, n_o(\vec{\sigma}_o)]^2 + \vec{K}^2(\tau,\vec{\sigma}_o)}.
\end{aligned} \tag{B6}$$

2) Let us now consider some cases of barotropic, $p = p(\rho(\hat{n},\hat{s}_o))$, and isentropic, $p = p(\rho(\hat{n}))$, fluids. Let us remember that the dominant energy condition on the energy-momentum tensor requires $|p| \leq \rho$.

2a) $p = k\,\rho(\hat{n})$ $(k \neq -1)$, whose equation of state is $\rho(\hat{n}) = \mu\,\hat{n}^{k+1}$. With $X = \hat{n}$ we get $\frac{\partial\rho}{\partial X} = (k+1)\,\mu\,X^k$ and Eq.(B1) becomes

$$[(k+1)\,\mu]^2\, X^{2(k-1)}\, (X^2 - B^2) = A. \tag{B7}$$

This equation can be solved in various cases, in particular for $k = 1$ and $k = \frac{1}{3}$

A) $k = 1$, $p = \rho$, $\rho = \mu\,\hat{n}^2$, we have:

$$X = \sqrt{B^2 + \frac{A}{4\mu^2}}, \tag{B8}$$

and then

$$\begin{aligned}
\mathcal{H}_\perp(\tau,\vec{\sigma}) &= \rho_\mu(\tau,\vec{\sigma})\, l^\mu(\tau,\vec{\sigma}) - \int d^3\sigma_o\, \delta^3(\vec{\sigma} - \vec{\Sigma}(\tau,\vec{\sigma}_o)) \Bigg[\frac{2 n_o(\vec{\sigma}_o)\,\mu}{\sqrt{\gamma(\Sigma)}\, \det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)} + \\
&+ \sqrt{\gamma(\Sigma)}\, \det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right) \mu \left(\frac{\gamma^{rs}(\Sigma) K_r(\tau,\vec{\sigma}_o) K_s(\tau,\vec{\sigma}_o)}{4\mu^2 n_o(\vec{\sigma}_o)} + \frac{n_o^2(\vec{\sigma}_o)\,\mu}{\sqrt{\gamma(\Sigma)} \det^2\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)} \right) \Bigg] \approx 0,\\
\mathcal{M} &= \int d^3\sigma_o\, \left[\frac{2 n_o(\vec{\sigma}_o)\,\mu}{\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)} + \det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right) \mu \left(\frac{\vec{K}^2(\tau,\vec{\sigma}_o)}{4\mu^2 n_o(\vec{\sigma}_o)} + \frac{n_o^2(\vec{\sigma}_o)\,\mu}{\det^2\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)} \right) \right]
\end{aligned} \tag{B9}$$

B) *Photon gas*, $k = \frac{1}{3}$, $p = \frac{1}{3}\rho$, $\rho = \mu\, \hat{n}^{4/3}$. Eq.(B1) is the following cubic equation in $Y = X^2$

$$\left(\frac{4\mu}{3}\right)^6 (Y - B^2)^3 - A^3\, Y^2 = 0. \tag{B10}$$

If we define

$$Z = Y - \frac{1}{3}\left(3B^2 + \left(\frac{3}{4\mu}\right)^6 A^3\right), \tag{B11}$$

to find the solution of the previous equation is equivalent to solve the cubic equation

$$Z^3 + C_1\, Z - C_o\, = 0, \tag{B12}$$

where

$$\begin{aligned} C_1 &= -2\left(\frac{3}{4\mu}\right)^6 A^3 B^2 - \frac{1}{3}\left(\frac{3}{4\mu}\right)^{12} A^6 \\ C_o &= \frac{2}{3}\left(\frac{3}{4\mu}\right)^{12} A^6 B^2 + \frac{2}{27}\left(\frac{3}{4\mu}\right)^{18} A^9 + \left(\frac{3}{4\mu}\right)^6 A^3 B^4, \end{aligned} \tag{B13}$$

Using the *Cardano solution* we obtain:

$$\begin{aligned} D &= \frac{C_o^2}{4} + \frac{C_1^3}{27} = \frac{1}{4}\left(\frac{3}{4\mu}\right)^{12} A^6 B^8 + \frac{1}{27}\left(\frac{3}{4\mu}\right)^{18} A^9 B^6, \\ Z &= \left(\frac{C_o}{2} + \sqrt{D}\right)^{\frac{1}{3}} + \left(\frac{C_o}{2} - \sqrt{D}\right)^{\frac{1}{3}}, \end{aligned} \tag{B14}$$

and then:

$$Y = X^2 = +\frac{1}{3}\left(3B^2 + \left(\frac{3}{4\mu}\right)^6 A^3\right) + \left(\frac{C_o}{2} + \sqrt{D}\right)^{\frac{1}{3}} + \left(\frac{C_o}{2} - \sqrt{D}\right)^{\frac{1}{3}}. \tag{B15}$$

We can obtain the explicit expression of the constraint $\mathcal{H}_\perp(\tau, \vec{\sigma}) \approx 0$ and of the invariant mass $\mathcal{M}$ using the solution (B15) in the eqs.(4.9) and (4.15). Since we have

$$\begin{aligned} &\rho = \mu\, X^{\frac{4}{3}} = \mu\, Y^{\frac{2}{3}}, \quad \frac{1}{X}\frac{\partial \rho}{\partial X} = \frac{4}{3}\mu\, X^{-\frac{2}{3}} = \frac{4}{3}\mu\, Y^{-\frac{1}{3}}, \\ &p(X) = X\frac{\partial \rho}{\partial X} - \rho = \frac{1}{3}\mu\, X^{\frac{4}{3}} = \frac{1}{3}\mu\, Y^{\frac{2}{3}}, \end{aligned} \tag{B16}$$

we obtain ($Y = X^2$)

$$\mathcal{M} = \int d^3\sigma_o \, \frac{\mu}{3} \, Y^{-\frac{1}{3}}(\tau, \vec{\sigma}_o) \, \det\left(\frac{\partial \Sigma}{\partial \sigma_o}\right) \Big[4B^2 - Y\Big] (\tau, \vec{\sigma}_o). \tag{B17}$$

2b) A variant is the equation of state $\rho(\hat{n}) = m\,\hat{n} + \frac{k}{\gamma-1}\,(m\,\hat{n})^\gamma$ ($\gamma \neq 1$) with $p = k\,(m\,\hat{n})^\gamma = (\gamma - 1)\,(\rho - m\,\hat{n})$. It is called a polytropic fluid ($\gamma = 1 + \frac{1}{n}$) by some authors and we can have $k = k(\hat{s}_o)$ in the non-isentropic case. Since we have

$$\frac{\partial \rho}{\partial X} = m\Big[1 + \frac{\gamma}{\gamma - 1}\,k\,(m\,X)^{\gamma-1}\Big], \tag{B18}$$

Eq.(B1) becomes

$$m^2\,(X^2 - B^2)\,\Big[1 + \frac{\gamma\,k\,m^{\gamma-1}}{\gamma - 1}\,X^{\gamma-1}\Big]^2 = A\,X^2. \tag{B19}$$

A) For $\gamma = 2$ we have the fourth order equation in X:

$$m^2\,(X^2 - B^2)\,\Big[1 + 2\,k\,m\,X\Big]^2 = A\,X^2. \tag{B20}$$

B) For $\gamma = 3$ we have a third order equation in $Y = X^2$:

$$m^2\,(Y - B^2)\,\Big[1 + \frac{3\,k\,m^2}{2}\,Y\Big]^2 = A\,Y. \tag{B21}$$

3) Standard polytropic perfect fluids have $p = k\,\rho^\gamma(\hat{n})$ ($\gamma = 1 + \frac{1}{n} \neq 1$) and

$$\rho(\hat{n}) = m\,\hat{n}\,\Big[1 - k\,(m\,\hat{n})^{\gamma-1}\Big]^{-\frac{1}{\gamma-1}}. \tag{B22}$$

Since we have

$$\frac{\partial \rho}{\partial X} = m\,\Big[1 - k\,(m\,\hat{n})^{\gamma-1}\Big]^{-\frac{\gamma}{\gamma-1}}, \tag{B23}$$

Eq.(B1) becomes

$$m^2\,\Big(1 - k\,m^{\gamma-1}\,X^{\gamma-1}\Big)^{-\frac{2\gamma}{\gamma-1}}\,(X^2 - B^2) = A\,X^2. \tag{B24}$$

For $\gamma = 3$ ($n = \frac{1}{2}$) it is a fourth order equation in $Y = X^2$:

$$m^2\,\Big(1 - k\,m^2\,Y\Big)^3\,(Y - B^2) = A\,Y. \tag{B25}$$

APPENDIX C: ON THE POISSON BRACKET

In Section III we observed that the constraints of the Hamiltonian formulation are not explicitly known as function of the canonical variables. Nevertheless it is possible to calculate their Poisson Bracket with a another functional on the phase space. To see this, we define the following short notations

$$
\begin{aligned}
A &= \frac{\gamma^{\check{r}\check{s}}(\Sigma) K_{\check{r}}(\tau, \vec{\sigma}_o) K_{\check{s}}(\tau, \vec{\sigma}_o)}{n_o^2(\vec{\sigma}_o)}, \\
B &= \frac{n_o(\vec{\sigma}_o)}{\det\left(\frac{\partial \Sigma}{\partial \sigma_o}\right) \sqrt{\gamma(\Sigma)}}, \\
Q &= \frac{1}{B}\left(\rho - X\,\frac{\partial \rho}{\partial X}\right) + \frac{B}{X}\frac{\partial \rho}{\partial X}, \\
P &= \frac{1}{B}\left(\rho - X\,\frac{\partial \rho}{\partial X}\right), \\
R &= \frac{X}{2B}\left(\frac{\partial \rho}{\partial X}\right)^{-1}.
\end{aligned} \tag{C1}
$$

With this notation the implicit definition of X, Eq.(4.9) can be rewritten in the form

$$
A = \left(\frac{\partial \rho}{\partial X}\right)^2 \left[-\frac{B^2}{X^2} + 1\right], \tag{C2}
$$

and the second constraint of Eqs.(4.10) is

$$
\mathcal{H}_\perp(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) l^\mu(\tau, \vec{\sigma}) - \mathcal{F}(\tau, \vec{\sigma}), \tag{C3}
$$

where

$$
\mathcal{F}(\tau, \vec{\sigma}) = \int d^3\sigma_o\, n_o(\vec{\sigma}_o)\, \delta^3(\vec{\sigma} - \vec{\Sigma}(\tau, \vec{\sigma}_o)) Q. \tag{C4}
$$

From Eq.(C2), using the rules on the Poisson Bracket, we can see that

$$
\begin{aligned}
\{., A\} = &-2B\left(\frac{\partial \rho}{\partial X}\right)^2 \frac{1}{X^2}\{., B\} + \\
&+ 2\left(\frac{\partial \rho}{\partial X}\right)\left[-\frac{\partial^2 \rho}{\partial X^2}\frac{B^2}{X^2} + \frac{\partial^2 \rho}{\partial X^2} + \frac{\partial \rho}{\partial X}\frac{B^2}{X^3}\right]\{., X\}.
\end{aligned} \tag{C5}
$$

Equally we can see that

$$\{.,\mathcal{F}(\tau,\vec{\sigma})\} = -\int d^3\sigma_o\, n_o(\vec{\sigma}_o)\, \frac{\partial}{\partial\sigma^{\check{r}}}\delta^3(\vec{\sigma}-\vec{\Sigma}(\tau,\vec{\sigma}_o))\, Q\, \{.,\Sigma^{\check{r}}(\tau,\vec{\sigma}_o)\} +$$

$$+\int d^3\sigma_o\, n_o(\vec{\sigma}_o)\, \delta^3(\vec{\sigma}-\vec{\Sigma}(\tau,\vec{\sigma}_o)) \times$$

$$\times \left[-\frac{1}{B^2}\left(\rho - X\,\frac{\partial\rho}{\partial X}\right) + \frac{1}{X}\,\frac{\partial\rho}{\partial X}\right]\{.,B\} +$$

$$+\int d^3\sigma_o\, n_o(\vec{\sigma}_o)\, \delta^3(\vec{\sigma}-\vec{\Sigma}(\tau,\vec{\sigma}_o)) \times$$

$$\times \left[\frac{1}{B}\left(-X\,\frac{\partial^2\rho}{\partial X^2}\right) + \frac{B}{X}\,\frac{\partial^2\rho}{\partial X^2} - B\,\frac{1}{X^2}\,\frac{\partial\rho}{\partial X}\right]\{.,X\}. \qquad \text{(C6)}$$

Using the results (C5) we can write

$$\left[\frac{1}{B}\left(-X\,\frac{\partial^2\rho}{\partial X^2}\right) + \frac{B}{X}\,\frac{\partial^2\rho}{\partial X^2} - B\,\frac{1}{X^2}\,\frac{\partial\rho}{\partial X}\right]\{.,X\} =$$

$$= -\frac{X}{2B}\left(\frac{\partial\rho}{\partial X}\right)^{-1}\left[\{.,A\} + 2B\left(\frac{\partial\rho}{\partial X}\right)^2\frac{1}{X^2}\{.,B\}\right]. \qquad \text{(C7)}$$

If we substitute this expression in Eq. (C6) and using the short notation (C1) we obtain

$$\{.,\mathcal{F}(\tau,\vec{\sigma})\} = -\int d^3\sigma_o\, n_o(\vec{\sigma}_o)\, \frac{\partial}{\partial\sigma^{\check{r}}}\delta^3(\vec{\sigma}-\vec{\Sigma}(\tau,\vec{\sigma}_o))\, Q\, \{.,\Sigma^{\check{r}}(\tau,\vec{\sigma}_o)\} +$$

$$+\int d^3\sigma_o\, \delta^3(\vec{\sigma}-\vec{\Sigma}(\tau,\vec{\sigma}_o))\, n_o(\vec{\sigma}_o)\, \left[-\frac{P}{B}\{.,B\} - R\{.,A\}\right]. \qquad \text{(C8)}$$

Being A and B known functions of the canonical variables, the previous equation permits to calculate the Poisson bracket with the constraint $\mathcal{H}_\perp(\tau,\vec{\sigma})$ although the X is unknown explicitly. The rule (C8) can be used, for example, to verify the algebra (4.13).

On the hyper-planes, the previous observations are again valid; in this case we have

$$A = -\frac{\vec{K}^2(\tau,\vec{\sigma}_o)}{n_o^2(\vec{\sigma}_o)}, \qquad B = \frac{n_o(\vec{\sigma}_o)}{\det\left(\frac{\partial\Sigma}{\partial\sigma_o}\right)}. \qquad \text{(C9)}$$

From Eq.(4.27) the invariant mass and the canonical generator of the internal boost are defined in term of the density

$$
\Delta(\tau, \vec{\sigma}_o) = \left[\det \left(\frac{\partial \Sigma}{\partial \sigma_o} \right) \left(\rho(X, s) - \frac{\partial \rho}{\partial X} X \right) + \frac{n_o^2(\vec{\sigma}_o)}{\det \left(\frac{\partial \Sigma}{\partial \sigma_o} \right)} \frac{1}{X} \frac{\partial \rho}{\partial X} \right] (\tau, \vec{\sigma}_o) =
$$

$$
= n_o(\vec{\sigma}_o)\, Q(\tau, \vec{\sigma}_o), \tag{C10}
$$

such that

$$
\mathcal{M} = \int d^3\sigma_o\, \Delta(\tau, \vec{\sigma}_o),
$$

$$
\vec{\mathcal{K}} = \int d^3\sigma_o\, \vec{\Sigma}(\tau, \vec{\sigma}_o)\, \Delta(\tau, \vec{\sigma}_o). \tag{C11}
$$

On them the previous rule becomes

$$
\{., \Delta(\tau, \vec{\sigma}_o)\} = n_o(\vec{\sigma}_o) \left[-\frac{P}{B} \{., B\} - R\{., A\} \right]. \tag{C12}
$$

This rule allows us to calculate the Poisson brackets of a functional on the phase space with the constraint also in the case of hyper-planes or Wigner hyper-planes. This is useful for example for getting the equations of motion from the Hamilton-Dirac equations.

APPENDIX D: GARTHENAUS-SCHWARTZ TRANSFORMATIONS

Using the notations of Section VI, let $\mathcal{G} = \vec{\mathcal{P}} \cdot \vec{\mathcal{Q}}$ be the canonical generator of a transformation, called *Garthenaus-Schwartz canonical transformations* [29]. If F is an arbitrary functions on the phase space, we have that its infinitesimal transformation is

$$\delta F = \delta\alpha \cdot \{F, \vec{\mathcal{P}} \cdot \vec{\mathcal{Q}}\}. \tag{D1}$$

For finite values of the parameter α the transformation can be written as

$$F(\alpha) = F + \int_0^\alpha d\overline{\alpha}\, \{F(\overline{\alpha}), \vec{\mathcal{P}}(\overline{\alpha}) \cdot \vec{\mathcal{Q}}(\overline{\alpha})\}. \tag{D2}$$

We are interested to the singular limit $\alpha \to \infty$ and we use the notation

$$F' = \lim_{\alpha\to\infty} F(\alpha). \tag{D3}$$

Deriving both sides of (D2) we can see that to realize the canonical transformation is equivalent to solve the differential equation

$$\begin{cases} \frac{dF}{d\alpha}(\alpha) &= \{F(\alpha), \vec{\mathcal{P}}(\alpha) \cdot \vec{\mathcal{Q}}(\alpha)\}, \\ \\ F(0) &= F. \end{cases} \tag{D4}$$

It is trivial to verify that

$$\vec{\mathcal{P}}(\alpha) = e^{-\alpha}\vec{\mathcal{P}} \;\Rightarrow\; \vec{\mathcal{P}}' = 0,$$

$$\vec{\mathcal{Q}}(\alpha) = e^{+\alpha}\vec{\mathcal{Q}} \;\Rightarrow\; \vec{\mathcal{Q}}' \to \infty. \tag{D5}$$

The usefulness of the singular limits is shown by the following observation [29]. Let F be a function on the phase space such that

$$\{\vec{\mathcal{P}}, F\} \equiv \{\vec{\mathcal{P}}(\alpha), F(\alpha)\} = 0. \tag{D6}$$

For this function the singular limit

$$\lim_{\alpha\to\infty} F(\alpha) = F', \tag{D7}$$

exists and is well defined. Moreover, if we define

$$\vec{G} = \{\vec{\mathcal{Q}}, F\}, \tag{D8}$$

the Jacoby identity implies that

$$\{\vec{\mathcal{P}}, \vec{G}\} = 0, \tag{D9}$$

and the limit

$$\lim_{\alpha \to \infty} \vec{G}(\alpha) = \vec{G}', \tag{D10}$$

exists and is well defined. In conclusion

$$\{\vec{\mathcal{P}}, F'\} = \lim_{\alpha \to \infty} \{\vec{\mathcal{P}}, F(\alpha)\} = \lim_{\alpha \to \infty} e^{+\alpha} \{\vec{\mathcal{P}}(\alpha), F(\alpha)\} = 0, \tag{D11}$$

because $\{\vec{\mathcal{P}}(\alpha), F(\alpha)\} \equiv 0$, and

$$\{\vec{\mathcal{Q}}, F'\} = \lim_{\alpha \to \infty} \{\vec{\mathcal{Q}}, F(\alpha)\} = \lim_{\alpha \to \infty} e^{-\alpha} \{\vec{\mathcal{Q}}(\alpha), F(\alpha)\} = 0, \tag{D12}$$

because the singular limit of $\{\vec{\mathcal{Q}}(\alpha), F(\alpha)\}$ is the well defined quantity $\vec{G}'$.

This observation can be applied to the relative variables $\Re^r(\tau, \vec{\sigma}_o), \wp^s(\tau, \vec{\sigma}_o)$. These variables are by construction such that

$$\{\vec{\mathcal{P}}, \Re^r(\tau, \vec{\sigma}_o)\} = \{\vec{\mathcal{P}}, \wp^s(\tau, \vec{\sigma}_o)\} = 0, \tag{D13}$$

and then their singular limits

$$\lim_{\alpha \to \infty} \Re^r(\alpha \mid \tau, \vec{\sigma}_o) = \Re'^r(\tau, \vec{\sigma}_o),$$

$$\lim_{\alpha \to \infty} \wp^s(\alpha \mid \tau, \vec{\sigma}_o) = \wp'^s(\tau, \vec{\sigma}_o), \tag{D14}$$

exist and are well defined. In particular we get

$$\{\vec{\mathcal{P}}, \Re'^r(\tau, \vec{\sigma}_o)\} = \{\vec{\mathcal{P}}, \wp'^r(\tau, \vec{\sigma}_o)\} = 0,$$

$$\{\vec{\mathcal{Q}}, \Re'^r(\tau, \vec{\sigma}_o)\} = \{\vec{\mathcal{Q}}, \wp'^r(\tau, \vec{\sigma}_o)\} = 0,$$

$$\{\Re'^r(\tau, \vec{\sigma}_o), \wp'^s(\tau, \vec{\sigma}'_o)\} = \delta^{rs}\, \delta^3(\vec{\sigma}_o - \vec{\sigma}'_o). \tag{D15}$$

In conclusion the coordinates of the table

$$\begin{array}{|c|c|}\hline \vec{\mathcal{Q}} & \Re'^{\,r}(\tau,\vec{\sigma}_o) \\ \vec{\mathcal{P}} & \wp'^{\,s}(\tau,\vec{\sigma}_o) \\ \hline \end{array}, \tag{D16}$$

are canonical coordinates. Moreover we can observe that

$$\begin{aligned}
\Re'^{\,r}(\tau,\vec{\sigma}_o) &= \Re^r(\tau,\vec{\sigma}_o) + \int_0^\infty d\alpha\ \{\Re^r(\alpha\mid\tau,\vec{\sigma}_o), \vec{\mathcal{P}}(\alpha)\cdot\vec{\mathcal{Q}}(\alpha)\} = \\
&= \Re^r(\tau,\vec{\sigma}_o) + \mathcal{P}^s \int_0^\infty d\alpha\ e^{-\alpha}\{\Re^r(\alpha\mid\tau,\vec{\sigma}_o), \vec{\mathcal{Q}}^s(\alpha)\} = \\
&= \Re^r(\tau,\vec{\sigma}_o) + \mathcal{P}^s\, I_{\Re}^{rs}, \\
\wp'^{\,r}(\tau,\vec{\sigma}_o) &= \wp^r(\tau,\vec{\sigma}_o) + \int_0^\infty d\alpha\ \{\wp^r(\alpha\mid\tau,\vec{\sigma}_o), \vec{\mathcal{P}}(\alpha)\cdot\vec{\mathcal{Q}}(\alpha)\} = \\
&= \wp^r(\tau,\vec{\sigma}_o) + \mathcal{P}^s \int_0^\infty d\alpha\ e^{-\alpha}\{\wp^r(\alpha\mid\tau,\vec{\sigma}_o), \vec{\mathcal{Q}}^s(\alpha)\} = \\
&= \wp^r(\tau,\vec{\sigma}_o) + \mathcal{P}^s\, I_{\wp}^{rs}.
\end{aligned} \tag{D17}$$

Due to the previous considerations, the $\{\Re^r(\alpha \mid \tau,\vec{\sigma}_o), \vec{\mathcal{Q}}^s(\alpha)\}$ and the $\{\wp^r(\alpha \mid \tau,\vec{\sigma}_o), \vec{\mathcal{Q}}^s(\alpha)\}$ are well defined constants in the singular limit. Then the integral $I_{\Re}, I_{\wp}$ in Eq.(D17) are well defined for the presence of $e^{-\alpha}$ factor. Then if we use explicitly the condition $\vec{\mathcal{P}} \approx 0$ we get

$$\Re'^{\,r}(\tau,\vec{\sigma}_o) \approx \Re^r(\tau,\vec{\sigma}_o) \qquad \wp'^{\,r}(\tau,\vec{\sigma}_o) \approx \wp^r(\tau,\vec{\sigma}_o). \tag{D18}$$

This is the case of the gauge fixing (6.27).

APPENDIX E: SOME SOLUTIONS FOR THE KERNEL Γ

In this Appendix we construct some kernels Γ that satisfy the conditions (6.5),(6.8) and (6.10) or Eq. (6.18).

The first solution is based on the possibility to read the kernel as distributions. For example, we can define in this case

$$\Gamma_K(\vec{\sigma}_o - \vec{\sigma}'_o) = \nabla^2_{\sigma_o}\, \delta^3(\vec{\sigma}_o - \vec{\sigma}'_o), \tag{E1}$$

and the second of Eqs. (6.5) is satisfied being reduced to

$$\nabla^2_{\sigma_o}\, 1 = 0. \tag{E2}$$

Let us define the usual symmetric Green function $c(\vec{\sigma}_o - \vec{\sigma}'_o)$

$$\nabla^2_{\sigma_o}\, c(\vec{\sigma}_o - \vec{\sigma}'_o) = \delta^3(\vec{\sigma}_o - \vec{\sigma}'_o), \tag{E3}$$

and let us make the following *ansatz* on Γ_Σ

$$\Gamma_\Sigma(\vec{\sigma}_o, \vec{\sigma}'_o) = c(\vec{\sigma}_o - \vec{\sigma}'_o) + f(\vec{\sigma}'_o). \tag{E4}$$

Then Eq. (6.10) is satisfied

$$\nabla^2_{\sigma_o}\, \Gamma_\Sigma(\vec{\sigma}_o, \vec{\sigma}'_o) = \nabla^2_{\sigma_o}\, c(\vec{\sigma}_o - \vec{\sigma}'_o) = \delta^3(\vec{\sigma}_o - \vec{\sigma}'_o). \tag{E5}$$

The function f is determined imposing Eq.(6.5) and we get

$$f(\vec{\sigma}'_o) = -\frac{1}{\mathcal{N}} \int d^3\sigma_{o1}\, n_o(\vec{\sigma}_{o1}) c(\vec{\sigma}_{o1} - \vec{\sigma}'_o). \tag{E6}$$

Automatically also Eq.(6.8) is satisfied. In conclusion

$$\Gamma_K(\vec{\sigma}_o, \vec{\sigma}'_o) = \nabla^2_{\sigma_o}\, \delta^3(\vec{\sigma}_o - \vec{\sigma}'_o),$$

$$\Gamma_\Sigma(\vec{\sigma}_o, \vec{\sigma}'_o) = c(\vec{\sigma}_o - \vec{\sigma}'_o) - \frac{1}{\mathcal{N}} \int d^3\sigma_{o1}\, n_o(\vec{\sigma}_{o1}) c(\vec{\sigma}_{o1} - \vec{\sigma}_o), \tag{E7}$$

is a distribution-like solution for the kernels Γ.

Another class of possible solutions is obtained if we use the representation $\Gamma^K_{\underline{n}}(\vec{\sigma}_o), \Gamma^\Sigma_{\underline{n}}(\vec{\sigma}_o)$ given by Eq. (6.17). Then we can consider a second base of orthonormal functions $\Psi_{\underline{n}}(\vec{\sigma}_o)$ such that

$$\int d^3\sigma_o \, n_o(\vec{\sigma}_o) \, \Psi_{\underline{o}}(\vec{\sigma}_o) \neq 0. \tag{E8}$$

The base $n_o(\vec{\sigma}_o), \Psi_{\underline{n}\neq\underline{o}}(\vec{\sigma}_o)$ is a complete, non orthonormal base of functions. Using the *Gram-Schmidt procedure* [34] we can construct the orthonormal base $\Psi'_{\underline{n}}(\vec{\sigma}_o)$ such that in particular

$$\Psi'_{\underline{o}}(\vec{\sigma}_o) = \frac{n_o(\vec{\sigma}_o)}{R}, \tag{E9}$$

with the normalization constant

$$R = \int d^3\sigma_o \, n_o^2(\vec{\sigma}_o). \tag{E10}$$

The other elements of $\Psi'_{\underline{n}\neq\underline{o}}(\vec{\sigma}_o)$ are given by the recurrence formula of the Gram-Schmidt's algorithm. With these definitions the first of the conditions (6.18) is satisfied if we choose

$$\Gamma^{\Sigma}_{\underline{o}}(\vec{\sigma}_o) = 0,$$

$$\Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_o) = \Psi'_{\underline{n}}(\vec{\sigma}_o) \quad if \quad \underline{n} \neq \underline{o}. \tag{E11}$$

The fourth of Eqs.(6.18) is satisfied if the Γ^K's have the following form

$$\Gamma^{K}_{\underline{o}}(\vec{\sigma}_o) = 0,$$

$$\Gamma^{K}_{\underline{n}}(\vec{\sigma}_o) = \Psi'_{\underline{n}}(\vec{\sigma}_o) - c_{\underline{n}}\Psi'_{\underline{o}}(\vec{\sigma}_o) \quad if \quad \underline{n} \neq \underline{o}. \tag{E12}$$

We use the second of Eqs.(6.18) for fixing the values of the coefficients $c_{\underline{n}}$

$$c_{\underline{n}} = -\frac{R}{\mathcal{N}} \int d^3\sigma_o \, \Psi'_{\underline{n}}(\vec{\sigma}_o) \quad \underline{n} \neq \underline{o}. \tag{E13}$$

With this choice also the third of Eqs.(6.18) is satisfied. In fact we can calculate explicitly the sum in the left-hand side of this conditions using the completeness of the basis $\Psi'_{\underline{n}}(\vec{\sigma}_o)$

$$\sum_{\underline{n}} \Gamma^{\Sigma}_{\underline{n}}(\vec{\sigma}_{1o})\Gamma^{K}_{\underline{n}}(\vec{\sigma}_{2o}) = \sum_{\underline{n}\neq\underline{o}} \Psi'_{\underline{n}}(\vec{\sigma}_{1o})\Psi'_{\underline{n}}(\vec{\sigma}_{2o}) + \sum_{\underline{n}\neq\underline{o}} c_{\underline{n}}\Psi'_{\underline{o}}(\vec{\sigma}_{2o})\Psi'_{\underline{n}}(\vec{\sigma}_{1o}) =$$

$$= \delta^3(\vec{\sigma}_{1o} - \vec{\sigma}_{2o}) - \Psi'_{\underline{o}}(\vec{\sigma}_{2o})\Psi'_{\underline{o}}(\vec{\sigma}_{1o}) + \sum_{\underline{n}\neq\underline{o}} c_{\underline{n}}\Psi'_{\underline{o}}(\vec{\sigma}_{2o})\Psi'_{\underline{n}}(\vec{\sigma}_{1o}) =$$

$$= \delta^3(\vec{\sigma}_{1o} - \vec{\sigma}_{2o}) + \Psi'_{\underline{o}}(\vec{\sigma}_{2o}) \sum_{\underline{n}} c_{\underline{n}}\Psi'_{\underline{n}}(\vec{\sigma}_{1o}), \tag{E14}$$

with $c_{\underline{o}} = -1$ in the last line . Finally we can observe that, in the sum, the $c_{\underline{n}}$ are the components on the base $\Psi'_{\underline{n}}(\vec{\sigma}_o)$ of the $(-R/\mathcal{N})$ constant function and then we have in accord with Eqs.(6.18)

$$\Psi'_{\underline{o}}(\vec{\sigma}_{2o}) \sum_{\underline{n}} c_{\underline{n}} \Psi'_{\underline{n}}(\vec{\sigma}_{1o}) = -\Psi'_{\underline{o}}(\vec{\sigma}_{2o}) \frac{R}{\mathcal{N}} = -\frac{n_o(\vec{\sigma}_{2o})}{\mathcal{N}}. \tag{E15}$$

In conclusion

$$\Gamma_\Sigma(\vec{\sigma}_{1o}, \vec{\sigma}_{2o}) = \sum_{\underline{n} \neq \underline{o}} \Psi'_{\underline{n}}(\vec{\sigma}_{1o}) \Phi_{\underline{n}}(\vec{\sigma}_{2o}),$$

$$\Gamma_K(\vec{\sigma}_{1o}, \vec{\sigma}_{2o}) = \sum_{\underline{n} \neq \underline{o}} \Phi_{\underline{n}}(\vec{\sigma}_{1o}) \Psi'_{\underline{n}}(\vec{\sigma}_{2o}) - \frac{n_o(\vec{\sigma}_{2o})}{\mathcal{N}} \sum_{\underline{n} \neq \underline{o}} c_{\underline{n}} \, \Phi_{\underline{n}}(\vec{\sigma}_{1o}). \tag{E16}$$

REFERENCES

[1] J.L.Tassoul, *Theory of Rotating Stars* (Princeton Univ. Press, Princeton, 1978).

[2] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science (Yeshiva University, New York, N.Y., 1964).

[3] L.Lusanna, *Towards a Unified Description of the Four Interactions in Terms of Dirac-Bergmann Observables*, invited contribution to the book *Quantum Field Theory: a 20th Century Profile* of the Indian National Science Academy, ed. A.N.Mitra (Hindustan Book Agency, New Delhi, 2000) (hep-th/9907081). Gen.Rel.Grav. **33**, 1579 (2001)(gr-qc/0101048).

[4] S.Adler and T.Buchert, Astron.Astrophys. **343**, 317 (1999) (astro-ph/9806320).

[5] J. D. Brown Class. Quantum Grav. **10**, 1579 (1993).

[6] L. Lusanna and D. Nowak-Szczepaniak, Int. J. Mod. Phys. **A15**, 4943 (2000).

[7] D.Bao, J.Marsden and R.Walton, Commun.Math.Phys. **99**, 319 (1985).
D.D.Holm, *Hamiltonian Techniques for Relativistic Fluid Dynamics and Stability Theory*, in *Relativistic Fluid Dynamics*, eds. A.Anile and Y.Choquet-Bruhat (Springer, Berlin, 1989).

[8] L. Lusanna, Int. J. Mod. Phys. **A12**, 645 (1997).

[9] D. Alba and L. Lusanna, Int. J. Mod. Phys. **A13**, 2791 (1998).

[10] H. Crater and L. Lusanna, Ann. Phys. **289**, 87 (2001) (hep-th/0001046).

[11] D. Alba, H. Crater and L. Lusanna, Int. J. Mod.Phys. **A16**, 3365 (2001) (hep-th/0103109).

[12] D. Alba and L. Lusanna, Int. J. Mod. Phys. **A13**, 3275 (1998).

[13] D. Alba, L. Lusanna and M. Pauri, J. Math. Phys. **43**, 373 (2002) (hep-th/0011014).

[14] D. Alba, L. Lusanna and M. Pauri, J. Math. Phys. **43**, 1677 (2002) (hep-th/0102087).

[15] D. Alba, L. Lusanna, M. Pauri, *Multipolar Expansions for the Relativistic N-Body Problem in the Rest-Frame Instant Form*(hep-th/0103092).

[16] W.G. Dixon, J. Math. Phys. **8**, 1591 (1967).

[17] W. G. Dixon, *Exended Bodies in General Relativity: Their Description and Motion*, Ed. J. Ehlers (North Holland, Amsterdam, 1979).

[18] L.Lusanna, *The Rest-Frame Instant Form of Metric Gravity*, Gen.Rel.Grav. **33**, 1579 (2001)(gr-qc/0101048).

[19] L.Lusanna and S.Russo, *A New Parametrization for Tetrad Gravity*, Gen.Rel.Grav. **34**, 189 (2002)(gr-qc/0102074).

[20] R.De Pietri, L.Lusanna, L.Martucci and S.Russo, *Dirac's Observables for the Rest-Frame Instant Form of Tetrad Gravity in a Completely Fixed 3-Orthogonal Gauge*, to appear in Gen.Rel.Grav. (gr-qc/0105084).

[21] J.Agresti, R.DePietri, L.Lusanna and L.Martucci, *Hamiltonian Linearization of the Rest-Frame Instant Form of Tetrad Gravity in a Completely Fixed 3-Orthogonal Gauge: a Radiation Gauge for Background-Independent Gravitational Waves in a Post-Minkowskian Einstein Space-Time*, in preparation.

[22] D.Alba, *Eulerian Coordinates for Non-Relativistic Perfect Fluids and the Ellipsoidal Equilibrium Configurations of Self-Gravitating Incompressible Fluids*, in preparation.

[23] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium*, Dover Publications Inc., New York (1968).

[24] A. J. Hanson and T. Regge, Ann. Phys. **87**, 498 (1974).

[25] P.A.M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[26] H.Leutwyler and J.Stern, Ann.Phys.(N.Y.) **112**, 94 (1978).

[27] M. Pauri and G. M. Prosperi, J. Math. Phys. **16**, 1503 (1975).

[28] C. Møller, Ann. Inst. Henri Poincaré, **11**,251 (1949); *The theory of Relativity* (Oxford University Press, Oxford, 1957).

[29] H. Osborn, Phys. Rev. **176**, 1514 (1968).

[30] R. G. Littlejohn and M. Reinsch, Rev. Mod. Phys. **69**, 213 (1997).

[31] A. Lucenti, L.Lusanna and M. Pauri, J. Phys. A31, 1633 (1998).

[32] H. Goldstein, *Classical Mechanics*, Addison-Wesley (1959).

[33] L. D. Landau and E. M. Lifsits, *Meccanica*, Editori Riuniti (1976).

[34] H. Hochstadt, *The functions of the mathematical physics*, Wiley-Interscience (1971).

[35] R. Arnowitt, S. Deser, C. W. Misner, *The dynamics of General relativity* in L. Witten, *Gravitation: an introduction to modern research*, Ed. Wiley (1962)